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SOME p-GROUPS OF MAXIMAL CLASS(1)

BY
R. J. MIECH

ABSTRACT. This paper deals with the construction of some p-groups of maximal class.

Introduction. Let G be a group, $G_2 = [G, G]$ be the commutator subgroup of G and $G_{i+1} = [G_i, G]$ for $i \geq 2$. A p-group G of order p^n is said to be of maximal class if $|G\colon G_2| = p^2$ and $|G_i\colon G_{i+1}| = p$ for $i = 1, \ldots, n-1$. In addition if G is of maximal class then G_1 is defined to be the largest subgroup of G such that $[G_1, G_2] \leq G_4$ and, in this paper, w is defined to be the smallest integer such that G_w is abelian. The general theory of these groups can be found in [3, p. 361].

This paper deals with the construction of the p-groups of maximal class and order p^n where $p \geq 5$, G_1 is of class 2, and (roughly) $n \geq 2p$, $w \leq n/4$. We take $p \geq 5$ for the groups under consideration are known when $p = 2$ or $p = 3$. When $p = 2$, G_1 is cyclic, so G is metacyclic; when $p = 3$, G_2 is abelian, that is G is metabelian, so these groups are also easy to classify [1, p. 82]. The structure of the groups is quite different when $p \geq 5$. If G is of order p^n and G_w is the first abelian member of the lower central series for G then, as we shall see, w may be as large as $n/4$. The other assumptions, G_1 is of class 2 and $n \geq 2p$, are made to keep commutator calculations from getting out of hand.

The following conventions, assumptions, and results will be used throughout this paper: G is a p-group of maximal class and order p^n where $n \geq p+2$; $G_1, G_2, \ldots$ are defined as above; w is the smallest positive integer such that G_w is abelian; G is generated by s and s_1 where $s \in G - G_1$ and $s_1 \in G_1 - G_2$; $s_i = [s_{i-1}, s]$ for $i = 2, 3, \ldots, n$; $G_i = \langle s_i, G_{i+1}\rangle$ for $i = 1, 2, \ldots, n-1$; $s_m = 1$ for $m \geq n$. The symbol $a(i, \nu)$ will always denote an integer such that $0 \leq a(i,\nu) \leq p-1$.

We shall also assume that $w \geq 3$. Those groups where $w \leq 2$ and G_1 is of class 2 have been classified [4].

The starting point of this paper is the known result: If $w \geq 3$ then $[G_{w-1}, G_w] \leq G_{n-p+3}$. This was proved by Blackburn [1, p. 77]; it will also be proved here for it follows quite easily from results we have and need for other purposes. The relation $[G_{w-1}, G_w] \leq G_{n-p+3}$ is equivalent to

$$[s_{w-1}, s_w] = \prod_{q=0}^{p-4} s_{N+q}^{a(w-1,N+q)}$$

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where $N = n - p + 3$ and, since G_{w-1} is not abelian, at least one of the $a(w-1, n+q)$ is not zero. Once this has been established we prove

Theorem 1. *Let G be a p-group of maximal class and order p^n with G_1 of class* 2. *Suppose that* $3 \leq w \leq (n - 2p + 12)/4$. *Then, for* $i = w - 1, w - 2, \ldots, 1$,

$$[s_i, s_{i+1}] = \prod_{q \geq 0} s_{N(i)+q}^{a(i, N(i)+q)}$$

where $N(i) = n - 2(w - i) - p + 5$ *and the* $a(i, \nu)$ *are integers such that* $0 \leq a(i, \nu) \leq p - 1$.

The equation of Theorem 1 is one of the defining relations of our groups. There is another set of relations involving double products of commutators. We single out these products in

Theorem 2. *Let G be as in Theorem* 1. *Set*

$$\tau(i, \nu) = [s_i, s_{i+1}, \overbrace{s, \ldots, s}^{\nu}].$$

Let

$$B(i) = \prod_{j=0}^{p-2} \prod_{\ell=0}^{p-2-j} \tau(i+j, \ell)^{(-1)^j \binom{\ell+j}{j} \binom{p+j}{\ell+2j+2}},$$

$$\beta(i, 0) = \prod_{j=0}^{p-3} \prod_{\ell=0}^{p-3-j} \tau(i+j, \ell)^{(-1)^{j+1} \binom{\ell+j+1}{j+1} \binom{p+j}{\ell+2j+3}},$$

$$\beta(i, 1) = \prod_{j=0}^{p-2} \prod_{\ell=0}^{p-2-j} \tau(i+j, \ell)^{(-1)^{j+1} \binom{\ell+j+1}{j+1} \binom{p+j}{\ell+2j+2}}.$$

Then $B(i) = \beta(i, 1) = 1$ *for* $i = 1, 2, \ldots, w - 1$; $\beta(i, 0) = 1$ *for* $i = 2, \ldots, w - 1$.

Some comments about these products might be in order. First of all they are connected by the equation

$$[\beta(i, 0), s]\beta(i, 1)^{-1} = B(i),$$

and there are other relations between them. The basic relations turn out to be: $B(1) = 1$ and $\beta(i, 0) = 1$ for $i = 2, \ldots, w - 1$. Secondly the conditions $B(1) = 1$ and $\beta(i, 0) = 1$ for $i \geq 2$ are equivalent to a set of congruences involving the exponents $a(i, \nu)$ of Theorem 1. These congruences are rather complicated but it is not too difficult to see that, in them, the numbers

$$\begin{array}{l} a(1, n-p+2), a(1, n-p+3), \ldots, a(1, n-1) \\ \qquad\qquad a(2, n-p+3), \ldots, a(2, n-1) \\ \qquad\qquad\qquad \cdot \\ \qquad\qquad\qquad \cdot \\ \qquad\qquad\qquad \cdot \\ \qquad\qquad a(w-1, n-p+3), \ldots, a(w-1, n-1) \end{array}$$

can be considered as independent variables, subject to the condition that $a(w-1, \nu) \neq 0$ for some ν, and the remaining $a(i, \nu)$ are determined by this set.

There is one congruence in the above set that is simple. We mention it as an example:

Theorem 3. *Let G be as in Theorem* 1. *Set*

$$\vartheta(0) = 1, \qquad \vartheta(1) = -3,$$

$$\vartheta(k) = -\frac{1}{(2k-1)}\binom{2k-1}{k}, \qquad k = 2, \ldots, p-2.$$

Then for $j \geq 0$ and $j \equiv k \bmod (p-1)$ we have

$$a(w-1-j, N(w-1-j)) \equiv \vartheta(k) a(w-1, N(w-1)) \bmod p.$$

Thus (see Theorem 1) the "leading" exponent of $[s_{w-1-j}, s_{w-j}]$ is a multiple of the leading exponent of $[s_{w-1}, s_w]$. Note that if $a(w-1, N(w-1)) = 0$ then $a(w-1-j, N(w-1-j)) = 0$ for all j. However if $a(w-1, N(w-1)) \neq 0$ it does not follow that $a(w-1-j, N(w-1-j)) \neq 0$ for all j, since $\vartheta(j) \equiv 0 \bmod p$ for $j = (p+3)/2, \ldots, p-2$ when $p \geq 7$.

Theorems 1 and 3 suggest a number δ defined by:

$$a(i, N(i) + q) = 0 \qquad \text{for } q = 0, \ldots, \delta - 1, i = w-1, w-2, \ldots, 1,$$

$$a(i, N(i) + \delta) \neq 0 \qquad \text{for } \textit{some } i.$$

Since $[s_{w-1}, s_w] \neq 1$ we have $0 \leq \delta \leq p-4$. The number δ plays an important role here. For, first, if δ is fixed then the inequality on w in Theorem 1 can be extended to $3 \leq w \leq (n - 2p + 2\delta + 12)/4$. Secondly δ is an invariant of the group for G has a degree of commutativity $k = n - 2w - p + 4 + \delta$. That is $[G_i, G_j] \leq G_{i+j+k}$ for all $i, j \geq 1$.

Incidentally the upper bound on w, $w \leq (n - 2p + 2\delta + 12)/4$, insures that if $[s_i, s_{i+1}]$ is defined by

$$[s_i, s_{i+1}] = \prod_{q \geq \delta} s_{N(i)+q}^{a(i, N(i)+q)}$$

then G_1 is of class 2. There might be groups where G_1 is of class 2 and w does not satisfy the inequality, but their existence would depend on particular and peculiar values assumed by the parameters $a(i, \nu)$. Further details about this can be found toward the end of §1; see Lemma 1.17.

One other product must be defined. Let

$$U(j, \ell) = \sum_{r=1}^{j+1} \binom{2r-1}{r}\binom{j+r+1}{2r}\binom{p}{j+\ell+r+2}$$

and let $\beta(i, 0)$ be defined as in Theorem 2. Set

$$q(i) = \beta(i, 0) \prod_{j=0}^{p-2} \prod_{\ell=0}^{p-2-j} \tau(i+j, \ell)^{-U(j, \ell)}.$$

The product $q(i)$ arises when we consider pth powers in G.

We can now state the main result of this paper:

Theorem 4. *Let p be a prime with $p \geq 5$. Let n, w and δ be integers such that*:

$$n \geq p + 2, \qquad n \geq 2p - 2\delta,$$
$$0 \leq \delta \leq p - 4, \qquad 3 \leq w \leq (n - 2p + 2\delta + 12)/4.$$

Let G be a p-group of maximal class and order p^n generated by s and s_1 with $s_i = [s_{i-1}, s]$ for $i \geq 2$. Suppose G_1 is of class 2 and G_w is the first abelian member of the lower central series for G. Then:

(a) *For some integer y, $s^p = s_{n-1}^y$.*

(b) *For $i = 1, 2, \dots, w-1$, $[s_i, s_{i+1}] = \prod_{q \geq \delta} s_{N(i)+q}^{a(i,N(i)+q)}$ where $N(i) = n - 2(w - i) - p + 5$.*

(c) *For $B(1)$ and $\beta(i,0)$ defined as in Theorem 2, $B(1) = 1$, $\beta(i,0) = 1$, $i = 2, 3, \dots, w-1$.*

(d) *For $i = 1, 2, \dots, n-1$,*

$$s_i^{\binom{p}{1}} \dots s_{i+p-1}^{\binom{p}{p}} q(i) = s_{n-1}^{x(i)}$$

where $q(i)$ is defined as above, $x(1)$ is some integer, and $x(i) = 0$ for $i \geq 2$.

Conversely given any n, w, δ and set of integers $\{a(i,\nu)\}$ satisfying these conditions there is a group of maximal class and order p^n defined by these relations.

The conditions $s_i = [s_{i-1}, s]$ and $s^p = s_{n-1}^y$ of Theorem 4 are from the general theory.

It is not difficult to determine an upper bound for the number of groups of the form given in Theorem 4. The independent parameters are, for fixed n and w : y, from part (a); $x(1)$, from part (d); and, from part (c)

$$\begin{aligned}
&a(1, n-p+2), a(1, n-p+3), \dots, a(1, n-1)\\
&\qquad a(2, n-p+3), \dots, a(2, n-1)\\
&\qquad\qquad \vdots\\
&\qquad a(w-1, n-p+3), \dots, a(w-1, n-1)
\end{aligned}$$

where $a(w-1, \nu) \neq 0$ for some ν. Since each of these parameters, subject to the last condition, can vary from 0 to $p - 1$ there are at most $(p^{p-3} - 1)p^{(p-3)(w-2)+3}$ such groups.

Since this paper is rather long an outline of its structure might be useful. We list and give a brief description of its parts in what follows.

1. This section begins with several lemmas on commutators and powers and then goes on to some consequences of these results. We have:

Lemma 1.2, a result that is used to express commutators of the form $[s_i, s_{i+t}]$ in terms of the commutators $[s_i, s_{i+1}, s, \ldots, s]$.

Lemmas 1.3 and 1.4, several results on pth powers.

Lemma 1.6, which introduces the product $B(i)$.

Lemmas 1.7 to 1.9, a proof of Theorem 1.

Lemmas 1.10 to 1.13, a proof of Theorem 2.

Lemmas 1.14 to 1.16 show that the condition $B(i) = 1$ is equivalent to a system of congruences on the exponents $a(i, v)$.

Lemma 1.17 and the balance of §1 concern the degree of commutativity of G and the upper bound for w assumed in Theorem 4.

2. This section is about the products $\beta(i, 0)$ and $\beta(i, 1)$. The main results are:

Lemma 2.1, a reduction of $\beta(i, 0)$ and $\beta(i, 1)$ to the form given in Theorem 2.

Lemma 2.2, a translation of the condition $\beta(i, 0) = 1$ to a system of congruences on the $a(i, v)$.

Lemma 2.3 gives a connection between $\beta(i, 0)$, $\beta(i, 1)$ and $B(i)$ which leads to the fact that we get $\beta(i, 0) = 1$ by choosing the parameter $a(i, n - p + 2)$ in a suitable way.

Lemma 2.4, a result for reducing $[s_i, s_{i+1}]^p$ to commutators of the form $[s_{i+1}, s_{i+1+k}]$ where $k \geq 1$.

Lemmas 2.7 and 2.8 establish a connection between $q(i)$ of Lemma 1.3 and the products $\beta(i, 0)$ and $\beta(i, 1)$.

Lemmas 2.9 to 2.11 lead to an equation between $q^{-1}(i)r(i)$ of Lemma 1.5 and the products $\beta(i, 1)$ and $\beta(i, 0)$.

3. This section contains an isolated part of the extension argument. The main result, Lemma 3.11, shows that if $B(i) = 1$ then $\beta(i, 1) = 1$. The proof of Lemma 3.11 is based on Lemma 2.4 and several complicated binomial identities. The identities are given in Lemmas 3.1 and 3.7.

4. We bring all our results together and prove that the groups under consideration exist.

1. This section contains several results about commutators, powers, and relations between them. We begin with a recursion formula.

Lemma 1.1. *Let G be a group of maximal class with G_1 of class 2. Set*

$$\lambda(i, t, v) = [s_i, s_{i+t}, \overbrace{s, \ldots, s}^{v}]$$

for $v \geq 0$. Then for $t \geq 1$

$$\lambda(i, t + 1, 0) = \lambda(i, t, 1)\lambda(i + 1, t - 1, 0)^{-1}\lambda(i + 1, t, 0).$$

Proof. We have

$$\lambda(i, t+1, 0) = [s_i, s_{i+t+1}] = s_{i+t+1}^{-s_i} s_{i+t+1} = [s, s_{i+t}]^{s_i} s_{i+t+1}.$$

In addition

$$[s, s_{i+t}]^{s_i} \overset{\bullet}{=} [s s_{i+1}^{-1}, s_{i+t}[s_{i+t}, s_i]].$$

So, employing the standard commutator relations and using the fact that G_1 is of class 2 one gets the equation of Lemma 1.1.

Repeated applications of Lemma 1.1 yield

Lemma 1.2. *Let G be a group of maximal class with G_1 of class 2 and suppose G_w is the first abelian member of the lower central series for G. Let $\lambda(i, t, \nu)$ be defined as above and set*

$$\tau(i, \nu) = [s_i, s_{i+1}, \overbrace{s, \ldots, s}^{\nu}].$$

Then for $i = w-1, w-2, \ldots, 1$ we have

$$\lambda(i, t, 0) = \prod_{j=0}^{w-1-i} \prod_{\omega=0}^{j} \tau(i+j, t-j-1-\omega)^{c(t,j,\omega)}$$

where $c(t,j,\omega) = (-1)^j \binom{j}{\omega}\binom{t-1-\omega}{j}$.

This is proved by induction on i and t. Note that since G_w is abelian we have

$$\lambda(i, t, 0) = [s_i, s_{i+t}] = 1 = [s_i, s_{i+1}] = \tau(i, 0)$$

when $i \geq w$. Thus the induction starts at $i = w-1$.

When $i = w-1$ we have, by Lemma 1.1, $\lambda(w-1, t+1, 0) = \lambda(w-1, t, 1) = [\lambda(w-1, t, 0), s]$. When $t = 1$ we have

$$\lambda(w-1, 2, 0) = [\lambda(w-1, 1, 0), s] = \tau(w-1, 1).$$

So, by induction on t,

$$\lambda(w-1, t, 0) = \tau(w-1, t-1).$$

Suppose next that the equation of Lemma 1.2 holds for $i+1, i+2, \ldots, w-1$. Then: (1) verify the given equation for $\lambda(i, 1, 0)$ and $\lambda(i, 2, 0)$ by direct computation; (2) assume the relation holds up to t; (3) apply the induction hypothesis to the right-hand side of

$$\lambda(i, t+1, 0) = \lambda(i, t, 1)\lambda(i+1, t-1, 0)^{-1}\lambda(i+1, t, 0)^{-1}.$$

Doing so one gets Lemma 1.2.

The next two lemmas are consequences of results in [5].

Lemma 1.3. *Suppose G is a group of maximal class with G_1 of class 2. Set*

$$b(k,\ell,t) = \binom{\ell+1}{t}\binom{k+t}{\ell+1} + \binom{k+2t-\ell-t}{t-1}\binom{k+t}{k+2t-\ell},$$

$$B(p,k,\ell) = \sum_{t=0}^{\ell+1} b(k,\ell,t)\binom{p}{k+1+t},$$

$$q(i) = \prod_{\ell=0}^{p-2}\prod_{k=\ell+1}^{p-1}[s_{i+\ell}, s_{i+k}]^{-B(p,k,\ell)}.$$

Then, for $i = 1, 2, \ldots, n-1$,

$$s_i^{\binom{p}{1}} \ldots s_{i+p-1}^{\binom{p}{p}} q(i) = s_{n-1}^{x(i)}$$

where $x(1)$ is an integer and $x(i) = 0$ for $i = 2, 3, \ldots, n-1$.

Proof. Take $x = s$ and $y = s_i$ in Theorem 4 of [5]. Then $\sigma(0) = y = s_i$, $\sigma(k) = s_{i+k}$ and we have

$$(ss_i)^p = s^p s_i^{\binom{p}{1}} \ldots s_{i+p-1}^{\binom{p}{p}} \prod_{\ell=0}^{p-2}\prod_{k=\ell+1}^{p-1}[s_{i+\ell}, s_{i+k}]^{-B(p,k,\ell)}.$$

Since s^p and $(ss_1)^p$ are in the center of G and $s^p = (ss_i)^p$ for $i \geq 2$ [3, p. 36] the assertion follows from this equation.

Lemma 1.4. *Let G be as above and suppose that $w \geq 3$. Set*

$$a(k,\ell,t) = \binom{\ell}{t}\binom{k+t}{\ell} - \binom{k+2t-\ell-1}{t-1}\binom{k+t}{k+2t-\ell},$$

$$A(p,k,\ell) = \binom{p}{k}\binom{p}{\ell} - \sum_{t=0}^{\ell} a(k,\ell,t)\binom{p}{k+t},$$

$$r(i) = \prod_{\ell=0}^{p-2}\prod_{k=\ell+1}^{p-1}[s_{i+\ell}, s_{i+k}]^{-A(p,k+1,\ell+1)}.$$

Then, for $i = 2, 3, \ldots, n-1$,

$$s_i^{\binom{p}{1}} \ldots s_{i+p-1}^{\binom{p}{p}} r(i) = 1.$$

Proof. Take $x = s$ and $y = s_{i-1}$ in Theorem 3 of [5]. Then $\sigma(0) = s_{i-1}$, $\sigma(k) = s_{i-1+k}$, and we have

$$\begin{aligned} s_{i-1}^{s^p} &= s_{i-1} s_i^{\binom{p}{1}} \ldots s_{i+p-1}^{\binom{p}{p}} \prod_{k=2}^{p-1}\prod_{\ell=1}^{k-1}[s_{i-1+k}, s_{i-1+\ell}]^{A(p,k,\ell)} \\ &= s_{i-1} s_i^{\binom{p}{1}} \ldots s_{i+p-1}^{\binom{p}{p}} \prod_{\ell=0}^{p-3}\prod_{k=\ell+1}^{p-2}[s_{i+\ell}, s_{i+k}]^{-A(p,k+1,\ell+1)}. \end{aligned}$$

Since s^p is in the center of G, Lemma 1.4 follows from this. (The range of multiplication in the last double product can be extended to $\ell = p-2$, $k = p-1$ since for $k = p-1$, $A(p,p,\ell+1) = 0$.)

Comparing Lemmas 1.3 and 1.4 and manipulating binomial coefficients one gets

Lemma 1.5. *Suppose $w \geq 3$ and $i \geq 2$. Set*

$$d(k,\ell,t) = 2\binom{\ell+t}{t}\binom{k+t}{\ell+1} + \binom{\ell}{t}\binom{k+t}{\ell},$$

$$D(p,\ell,k) = -\binom{p}{\ell+1}\binom{p}{k+1} + \sum_{t=0}^{\ell+1} d(k,\ell,t)\binom{p}{k+\ell+t}.$$

Then

$$q^{-1}(i)r(i) = \prod_{\ell=0}^{p-2}\prod_{k=\ell+1}^{p-1} [s_{i+\ell}, s_{i+k}]^{D(p,\ell,k)} = 1.$$

The function $B(i)$ that is introduced in the next lemma turns out to be one of the basic products connected with our groups.

Lemma 1.6. *Suppose G is a group of maximal class with G_1 of class 2. Let, for $i \geq 1$,*

$$B(i) = [s_i, s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}}].$$

Then $B(i) = 1$. Moreover,

$$B(i) = \prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j} \tau(i+j,\ell)^{k(j,\ell)}$$

where $k(j,\ell) = (-1)^j\binom{\ell+j}{j}\binom{p+j}{\ell+2j+2}$.

Proof. According to Lemma 1.3, $s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}} \quad q(i) = s_i^{-p} s_{n-1}^{x(i)}$. Since G_1 is of class 2 this implies that

$$B(i) = [s_i, s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}}] = 1.$$

As for the second part of Lemma 1.6: Using the fact that G_1 is of class 2 and then applying Lemma 1.2 we have

$$\begin{aligned} B(i) &= \prod_{t=1}^{p-1} \lambda(i,t,0)^{\binom{p}{t+1}} \\ &= \prod_{t=1}^{p-1}\prod_{j=0}^{w-1-i}\prod_{\omega=0}^{j} \tau(i+j, t-j-1-\omega)^{\binom{p}{t+1}c(t,j,\omega)} \\ &= \prod_{j=0}^{w-1-i}\prod_{\ell=0}^{p-2} \tau(i+j,\ell)^{k(j,\ell)} \end{aligned}$$

where

$$\begin{aligned} k(j,\ell) &= \sum_{t,\omega;\, t-\omega-j-1=\ell} c(t,j,\omega)\binom{p}{t+1} \\ &= \sum_{t=\ell+j+1}^{p-1} (-1)^j \binom{j}{t-\ell-j-1}\binom{\ell+j}{j}\binom{p}{t+1} \\ &= (-1)^j \binom{\ell+j}{j}\binom{p+j}{\ell+2j+2}. \end{aligned}$$

Finally, since $\tau(i,0) = [s_i, s_{i+1}] = 1$ for $i \geq w$ and $k(j,\ell) = 0$ for $\ell \geq p-2-j$ the range of j and ℓ in the product defining $B(i)$ can be altered to the one stated in Lemma 1.6.

We are now in a position to establish most of Theorem 1. To begin we have

Lemma 1.7. *Let G be a group of maximal class with G_1 of class 2. Suppose G_{w-1} is not abelian, G_w is abelian, and $w \geq 3$. Set*

$$[s_{w-1}, s_w] = \prod_{q=0}^{m-1} s_{n-m+q}^{a(n-m+q)}.$$

Then $1 \leq m \leq p-3$ and $0 < a(n-m) < p$.

Proof. First, since G_{w-1} is not abelian and G_w is we have $[s_{w-1}, s_w] \neq 1$. Thus for some $m \geq 1$, $a(n-m) \neq 0$.

We need to bring a few facts together to get the upper bound on m, $m \leq p-3$. First take $i = w-1$ in Lemma 1.5 to get

$$\prod_{k=1}^{p-1} [s_{w-1}, s_{w-1+k}]^{k\binom{p+1}{k+2}} = 1. \tag{1}$$

Next, by Lemma 1.2,

$$[s_{w-1}, s_{w-1+k}] = \prod_{q\geq 0} s_{n-m+q+k-1}^{a(n-m+q)}. \tag{2}$$

Finally for any $i \geq 1$ [1, p. 67],

$$s_i^p \equiv s_{i+p-1}^{-1} \bmod G_{i+p}.$$

Now, by (2) and (3),

$$[s_{w-1}, s_{w-1+k}]^p \in G_{n-m+k-1+p-1} \leq G_{n-m+p-2}$$

for all $k \geq 1$. Next, examining the term corresponding to $k = p-2$ in (1) we find that

$$[s_{w-1}, s_{w-1+p-2}]^{(p-2)(p+1)} \equiv s_{n-m+p-3}^{-2a(n-m)} \bmod G_{n-m+p-2}.$$

Finally for the last term in (1) we have

$$[s_{w-1}, s_{w-1+p-1}]^{p-1} \equiv 1 \bmod G_{n-m+p-2}.$$

Bringing these results together we have

$$s_{n-m+p-3}^{-2a(n-m)} \equiv 1 \bmod G_{n-m+p-2}.$$

Since $0 < a(n-m) < p$ this implies that $m \leq p-3$.

Lemma 1.7 is the first step in the inductive proof of

Lemma 1.8. *Set* $N(i) = n - 2(w-i) - p + 5$. *Then for* $i = w-1, w-2, \ldots, 1$ *we have*

$$[s_i, s_{i+1}] = \prod_{q \geq 0} s_{N(i)+q}^{a(i,N(i)+q)}.$$

The proof of Lemma 1.8 is based on

Lemma 1.9. *Let* $k(j, l)$ *be defined as in Lemma* 1.6 *and set*

$$B_j = \prod_{l=0}^{p-2-j} \tau(i+j, l)^{k(j,l)}.$$

Suppose that for $j \geq 0$

$$[s_{i+j}, s_{i+j+1}] = \prod_{q \geq 0} s_{N(i+j)+q}^{a(i+j,N(i+j)+q)}$$

where $N(i+j)$ *is a positive integer. Let* $y(j) = k(j, p-2-2j)a(i+j, N(i+j))$. *Then for* $j = 0, 1, \ldots, (p-3)/2$ *we have*

$$B_j \equiv s_{N(i+j)+p-2-2j}^{y(j)} \bmod G_{N(i+j)+p-1-2j}.$$

For $j = (p-1)/2, \ldots, p-2$ *we have* $B_j \in G_{N(i+j)}$.

Proof. We have

$$k(j, l) = (-1)^j \binom{l+j}{j} \binom{p+j}{l+2j+2} \equiv 0 \bmod p$$

for $l \leq p-3-2j$. Thus if $j \leq (p-3)/2$ then $B_j = B'_j B''_j$ where

$$B'_j = \prod_{l=0}^{p-3-2j} \tau(i+j, l)^{k(j,l)} \in G_{N(i+j)+p-1}$$

and

$$B''_j = \prod_{l=p-2-2j}^{p-2-j} \tau(i+j, l)^{k(j,l)} \equiv s_{N(i+j)+p-2-2j}^{y(j)} \bmod G_{N(i+j)+p-1-2j}.$$

The first part of Lemma 1.9 follows from these two relations; the second part is obvious.

Proof of Lemma 1.8. Assume, inductively, that $N(i+j) = n - 2(w-i-j) - p + 5$ for $j \geq 1$. Let $N(i)$ be the smallest positive integer such that $0 < a(i, N(i)) < p$.

Now by Lemma 1.6, $B(i) = B_0 \cdots B_{p-2} = 1$. By Lemma 1.9,

$$B_0 \equiv s_{N(i)+p-2}^{y(0)} \bmod G_{N(i)+p-1}.$$

Let $\pi = (p-2)/2$ and $n(i) = n - 2(w-i)$. Then by Lemma 1.9 and the induction hypothesis

$$B_1 \cdots B_\pi \equiv s_{n(i)+3}^{A} \bmod G_{n(i)+4}$$

where $A = \sum_{j=1}^{\pi} k(j, p-2-2j) a(i+j, N(i+j))$. Finally for $j \geq (p-1)/2$ we have $B_j \in G_{N(i+j)}$, and $N(i+j) = n - 2(w-i) + 2j - p + 5 \geq n(i) + 4$. Consequently

$$1 = B_0 B_1 \cdots B_{p-2} \equiv s_{N(i)+p-2}^{y(0)} s_{n(i)+3}^{A} \bmod G_h$$

where h is the minimum of $N(i) + p - 1$ and $n - 2(w-i) + 4$ and $y(0) = a(i, N(i))$. If we assume that $N(i) + p - 1 < n - 2(w-i) + 4$ we have $s_{N(i)+p-2}^{a(i,N(i))} \equiv 1 \bmod G_{N(i)+p-1}$. This implies that $N(i) + p - 1 < n - 2(w-i) + 4$ and $N(i) + p - 2 \geq n$ or $N(i) + p - 1 \geq n - 2(w-k) + 4$. Since we can assume $i \leq w - 2$ the first case is impossible. Thus $N(i) \geq n - 2(w-i) - p + 5$. If we now let the appropriate exponents be zero we may assume that $N(i) = n - 2(w-i) - p + 5$.

The above proof contains a relation for the exponents $a(i, N(i))$. We have

Lemma 1.10. *Let* $\pi = (p-3)/2$. *Then*

$$\sum_{j=0}^{\pi} (-1)^j \binom{p-2-j}{j} \binom{p+j}{j} a(i+j, N(i+j)) \equiv 0 \bmod p.$$

That is, if $\vartheta(0) = 1$, $\vartheta(1) = -3$, *and* $\vartheta(k) = -(1/(2k-1))\binom{2k-1}{k}$ *for* $k = 2, \ldots, p-2$ *then, for* $j \geq 0$ *and* $j \equiv k \bmod (p-1)$,

$$a(w-1-j, N(w-1-j)) \equiv \vartheta(k) a(w-1, N(w-1)) \bmod p.$$

Proof. Examining the proof of Lemma 1.8 and using the fact that $N(i) = n - 2(w-i) - p + 5 = n(i) - p + 5$ one sees that

$$1 = B(i) = s_{n(i)+3}^{S(i)} \bmod G_{n(i)+4}$$

where $S(i) = \sum_{j=0}^{\pi} k(j, p-2-2j) a(i+j, N(i+j))$. This gives us the first part of Lemma 1.10.

There are several steps to the proof of the second part of Lemma 1.10. The first is

Lemma 1.11. *Let* $A(i+j) = a(i+j)$, $N(i+j)$ *and*

$$S(i) = \sum_{j=0}^{(p-3)/2} (-1)^j \binom{p-2-j}{j}\binom{p+j}{j} A(i+j). \tag{1}$$

Then, for $q = 1, \ldots, p-2$,

$$S(w-1-q) \equiv \sum_{j=0}^{q} \binom{2j+1}{j} A(w-1-q+j) \bmod p. \tag{2}$$

Proof. We have $(-1)^j\binom{p-2-j}{j}\binom{p+j}{j} \equiv \binom{2j+1}{j} \bmod p$ when $0 \leq j \leq (p-3)/2$. In addition $A(w-1+t) \equiv 0 \bmod p$ for $t \geq 1$, so (1) reduces to (2) when $q \leq (p-3)/2$. Furthermore $\binom{2j+1}{j} \equiv 0 \bmod p$ when $(p-1)/2 \leq j \leq p-2$, so (1) is equivalent to (2) for $(p-1)/2 \leq q \leq p-2$.

Lemma 1.12. *Let* $K(x) = \binom{2x+1}{x}$ *and*

$$\vartheta(q) = \sum_{r=1}^{q} (-1)^r \sum_{x_1,\ldots,x_r \geq 1; x_1+\cdots+x_r=q} K(x_1)\cdots K(x_r).$$

Then, for $q = 1, \ldots, p-2$,

$$A(w-1-q) \equiv \vartheta(q) A(w-1) \bmod p.$$

Proof. This can be proved by induction on q. One has

$$S(w-1-q) \equiv A(w-1-q) + \sum_{j=1}^{q} K(j) A(w-1-q+j) \equiv 0 \bmod p,$$

which follows from (2) of Lemma 1.11 and Lemma 1.10.

Lemma 1.13. *Let*

$$\vartheta(r,q) = \sum_{x_1,\ldots,x_r \geq 1; x_1+\cdots+x_r=q} K(x_1)\cdots K(x_r)$$

and, as in Lemma 1.12, $\vartheta(q) = \sum_{r=1}^{q} (-1)^r \vartheta(r,q)$. *Then* $\vartheta(1) = -3$ *and*

$$\vartheta(q) = -\frac{1}{(2q-1)}\binom{2q-1}{q}$$

for $q \geq 2$.

Proof. To begin we have

$$(1+x)^{-3/2} = \sum_{k=0}^{\infty} \binom{-3/2}{k} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}\binom{2k+1}{k}(k+1)x^k.$$

Integrating this from 0 to t and then replacing t by $4t$ we get

$$\tfrac{1}{2}(1 - (1 + 4t)^{-1/2}) = \sum_{k=0}^{\infty} (-1)^k \binom{2k+1}{k} t^{k+1}.$$

That is

$$\frac{[(1 - 2t) - (1 + 4t)^{-1/2}]}{2t} = \sum_{k=1}^{\infty} (-1)^k \binom{2k+1}{k} t^k.$$

Let

$$\varphi(t) = \frac{1}{2t}[(1 - 2t) - (1 + 4t)^{-1/2}] = \sum_{k=1}^{\infty} (-1)^k \binom{2k+1}{k} t^k.$$

Note that $\varphi(t)^r = \sum_{q=r}^{\infty} (-1)^q \vartheta(r, q) t^q$. Thus

$$\sum_{r=1}^{\infty} (-\varphi(t))^r = \sum_{r=1}^{\infty} (-1)^r \sum_{q=r}^{\infty} (-1)^q \vartheta(r, q) t^q$$

and we have

$$\frac{-\varphi(t)}{1 + \varphi(t)} = \sum_{q=1}^{\infty} (-1)^q \vartheta(q) t^q.$$

But

$$\frac{-\varphi(t)}{1 + \varphi(t)} = -\tfrac{1}{2} + \tfrac{1}{2}(1 + 4t)^{1/2} + 2t = 3t + \sum_{q=2}^{\infty} (-1)^{q-1} \frac{1}{(2q-1)} \binom{2q-1}{q} t^q.$$

So, comparing coefficients we get Lemma 1.13.

We can now complete the proof of Lemma 1.10. First if $p = 5$ we have, by the first part of Lemma 1.10, $A(i + j) \equiv 2A(i + j + 1) \bmod 5$. Thus

$$A(w - 1 - k) \equiv 2^k A(w - 1) \bmod 5$$

for any $k \geq 0$. Since $\vartheta(k) \equiv 2^k \bmod 5$ for $k = 0, 1, 2, 3$ this completes the proof for the case $p = 5$.

Suppose then that $p \geq 7$. By Lemmas 1.11 to 1.13,

$$A(w - 1 - k) \equiv \vartheta(k) A(w - 1) \bmod p$$

for $k = 0, 1, \ldots, p - 2$ with $\vartheta(k)$ as given in Lemma 1.10. Since $\vartheta(k) \equiv 0 \bmod p$ for $k = (p + 3)/2, \ldots, p - 2$ we have, by the first part of Lemma 1.10 with $i = w - 1 - (p - 1)$ and $\pi = (p - 3)/2$,

$$A(w-1-(p-1)] + (-1)^{\pi}\binom{p-2-\pi}{\pi}\binom{p+\pi}{\pi}A(w-1-(p-1)+\pi)$$
$$\equiv 0 \bmod p.$$

Since $A(w-1-(p-1-\pi)) \equiv \vartheta((p+1)/2)A(w-1) \bmod p$ it follows that $A(w-1-(p-1)) \equiv A(w-1) \bmod p$. This, the first part of Lemma 1.10, and the fact that $\vartheta(k) = 0$ for $k = (p+3)/2, \ldots, p-2$ gives us the fact that the values of $A(i)$ repeat modulo $(p-1)$.

Lemma 1.10 illustrates the fact that the relation $B(i) = 1$ imposes certain conditions on the exponents $a(i,j)$. In this particular case we have $a(w-1, N(w-1))$ acting as an independent parameter while, for $i \gtrsim w-2$, the $a(i, N(i))$ play the role of dependent ones. The purpose of the next few lemmas is to derive a result that will enable us to determine which of the exponents act as independent parameters when we take $B(i) = 1$.

Lemma 1.14. *Let* $E(0) = -1$,

$$f(t_1, \ldots, t_r) = \frac{(-1)^{r+1}}{p^r}\binom{p}{t_1+1}\cdots\binom{p}{t_r+1},$$

and for $j \geq 1$

$$E(j) = \sum_{r=1}^{j} \sum_{I(r,j)} f(t_1, \ldots, t_r)$$

where $I(r,j) = \{1 \leq t_1, \ldots, t_r \leq p-2, t_1 + \cdots + t_r = j\}$. *Suppose that* G_i *is abelian. Then*

$$s_i^p = \prod_{j=0}^{n-p-i} s_{i+p-1+j}^{E(j)}.$$

Proof. Since G_i is abelian we have, by Lemma 1.3,

$$s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}} = 1.$$

That is

$$s_i^p = s_{i+p-1}^{-1} \prod_{t_1=1}^{p-2} (s_{i+t_1}^p)^{-f(t_1)}.$$

Lemma 1.11 follows from repeated applications of this formula.

Lemma 1.15. *Let* $B(i)$ *be defined as in Lemma* 1.6, $a(i, N(i)+q)$ *as in Lemma* 1.8 *and* $E(j)$ *as in Lemma* 1.14. *Set* $A(i,q) = a(i, N(i)+q)$ *and* $E(j) = 0$ *for* $j < 0$. *Let*

$$\vartheta(j,\ell) = \sum_{\nu=0}^{p-3-2j} E(\ell - 2j - \nu)\frac{(-1)^j}{p}\binom{\nu+j}{j}\binom{p+j}{\nu+2j+2},$$

$$F(i,r) = \sum_{j=0}^{p-2}\sum_{\ell=0}^{r} A(i+j,r-\ell)\frac{(-1)^j}{p}\binom{p-2-j+\ell}{j}\binom{p+j}{p+\ell},$$

$$G(i,r) = \sum_{j=0}^{\pi}\sum_{\ell=2j}^{r} A(i+j,r-\ell)\vartheta(j,\ell),$$

where $\pi = (p-3)/2$. *Let* $H(i,r) = F(i,r+1) + G(i,r)$ *for* $r \geq 0$. *Let* $n(i) = n - 2(w-i)$. *Then*

$$B(i) = s_{n(i)+3}^{F(i,0)} \prod_{r=0}^{2(w-i)-5} s_{n(i)+4+r}^{H(i,r)}.$$

Proof. By Lemma 1.6,

$$B(i) = \prod_{j=0}^{p-2}\prod_{\nu=0}^{p-2-j} \tau(i+j,\nu)^{k(j,\nu)}$$

where $k(j,\nu) \equiv 0 \bmod p$ for $\nu \leq p-3-2j$. So split $B(i)$ into two parts, $B(i) = P_1P_2$ where

$$P_1 = \prod_{j=0}^{p-2}\prod_{\nu=p-2-2j}^{p-2-j} \tau(i+j,\nu)^{k(j,\nu)}, \qquad P_2 = \prod_{j=0}^{\pi}\prod_{\nu=0}^{p-3-2j} \tau(i+j,\nu)^{k(j,\nu)}.$$

Recall that $\tau(i+j,0) = \prod_{q\geq 0} s_{N(i+j)+q}^{A(i+j,q)}$ where $N(i+j) = n - 2(w-i) + 2j - p + 5 = n(i) + 2j - p + 5$.

The product P_1 is easy to handle. We have

$$P_1 = \prod_{j=0}^{p-2}\prod_{\nu=p-2-2j}^{p-2-j}\prod_{q\geq 0} s_{N(i+j)+q+\nu}^{k(j,\nu)A(i+j,q)}.$$

Thus if we set $\nu = p-2-2j+\ell$, $\ell = 0,\ldots,j$, then $N(i+j)+\nu = n(i)+3+\ell$ and we have

$$P_1 = \prod_{r\geq 0} s_{n(i)+3+r}^{F(i,r)}$$

where $F(i,r) = \sum_{j=0}^{p-2}\sum_{\ell=0}^{r} A(i+j,r-\ell)k(j,p-2-2j+\ell)$.

The product P_2 is slightly more complicated since the exponents are divisible by p. To begin we have

$$P_2 = \prod_{j=0}^{\pi}\prod_{\nu=0}^{p-3-2j}\prod_{q\geq 0} s_{N(i+j)+\nu+q}^{A(i+j,q)k(j,\nu)} = \prod_{j=0}^{\pi}\prod_{x\geq 0} s_{N(i+j)+x}^{pG_1(i+j,x)}$$

where $G_1(i+j,x) = \sum_{\nu=0}^{x} A(i+j,x-\nu)k(j,\nu)/p$. Applying Lemma 1.14 here we get

$$P_2 = \prod_{j=0}^{\pi} \prod_{x\geq 0} \prod_{l\geq 0} s_{N(i+j)+p-1+x+l}^{E(l)G_1(i+j,x)} = \prod_{j=0}^{\pi} \prod_{y\geq 0} s_{n(i)+4+y+2j}^{G_2(i+j,y)}$$

where

$$\begin{aligned} G_2(i+j,y) &= \sum_{x=0}^{y} E(y-x)G_1(i+j,x) \\ &= \sum_{t=0}^{y} A(i+j,t) \sum_{v=0}^{y-t} E(y-t-v)\frac{k(j,v)}{p}. \end{aligned}$$

Finally

$$P_2 = \prod_{r\geq 0} s_{n(i)+r+4}^{G(i,r)}$$

where

$$\begin{aligned} G(i,r) &= \sum_{j=0}^{\pi} G_2(i+j,r-2j) \\ &= \sum_{j=0}^{\pi} \sum_{t=0}^{r-2j} A(i+j,t) \sum_{v=0}^{r-2j-t} E(r-2j-t-v)\frac{k(j,v)}{p} \\ &= \sum_{j=0}^{\pi} \sum_{l=2j}^{r} A(i+j,r-l) \sum_{v=0}^{l-2j} E(l-2j-v)\frac{k(j,v)}{p}. \end{aligned}$$

Since the v in the innermost sum must also satisfy the inequality $v \leq p-3-2j$ (see the definition of P_2) we get

$$G(i,r) = \sum_{j=0}^{\pi} \sum_{l=2j}^{r} A(i+j,r-l)\vartheta(j,l).$$

Lemma 1.15 follows from these results.

Lemma 1.16. *For $i = w-2, \ldots, 1$, the equation $B(i) = 1$ is equivalent to a system of congruences*

$$a(i,N(i)+j) \equiv \varphi(i,j) \bmod p, \qquad j = 0, \ldots, 2(w-i)-4$$

where $\varphi(i,j)$ is some function of $a(w-1,n-p+3), \ldots, a(w-1,n-1)$, and $a(k,n-p+2), \ldots, a(k,n-1)$, $k = w-2, \ldots, i+1$.

Proof. Suppressing most of the information appearing in the definitions of Lemma 1.15 we have

$$(1) \qquad H(i,r) = A(i,r+1) + \varphi(A(i,0), \ldots, A(i,r), \ldots, A(i+j,k), \ldots)$$

where φ is some function of $A(i,0), \ldots, A(i,r)$ as well as the $A(i+j,k)$, $j \geq 1$, $k \geq 0$. We also have

$$(2) \qquad B(i) = s_{n(i)+3}^{F(i,0)} \prod_{r=0}^{2(w-i)-5} s_{n(i)+4+r}^{H(i,r)} = 1.$$

So, by Lemma 1.14,

$$(3) \qquad H(i,r) + \varphi'[H(i,r-1),\ldots,H(i,0),F(i,0)] \equiv 0 \bmod p$$

where φ' is some function of the variables displayed. Putting (1) in (3) we see that (3) can be regarded as a defining congruence for $A(i,r+1)$. Since $A(i,r+1) = a(i,N(i)+r+1)$, $N(i) = n-2(w-i)-p+5$, and $r \leq 2(w-i)-5$ we have $N(i)+r+1 \leq n-p+1$. Thus we have defining congruences for the parameters $a(i,N(i)),\ldots,a(i,n-p+1)$. Lemma 1.13 follows from these results.

Lemma 1.16 is, of course, of limited utility. However we shall need it, and several analogues of it, to determine our course in the extension problem.

The next few lemmas deal with the upper bound for w in Theorem 4. To begin we have:

Lemma 1.17. *Let* $m(i,t) = n-2(w-i)-p+4+t$ *and*

$$e(i,t,r) = \sum_{j=0}^{w-i}\sum_{q=0}^{r}(-1)^j\binom{j}{r-q}\binom{t-1+r-q-j}{j}a(i+j,N(i+j)+q).$$

Then, for $t \geq 1$,

$$[s_i,s_{i+t}] = \prod_{r\geq 0} s_{m(i,t)+r}^{e(i,t,r)}.$$

This follows from Lemmas 1.2 and 1.8.

To continue, under our present assumptions we may have $a(i,N(i)) = 0$ for all i; likewise we might have $a(i,N(i)+1) = 0$ for all i. So let us define an integer δ by the conditions:

$$a(i,N(i)+q) = 0, \qquad 1 \leq i \leq w-1,\ 0 \leq q \leq \delta - 1;$$
$$a(i,N(i)+\delta) \neq 0, \qquad \text{for some } i.$$

Note that since $[s_{w-1},s_w] \neq 1$ we have $0 \leq \delta \leq p-4$. Furthermore, in Lemma 1.17, we have $e(i,t,r) = 0$ for $r < \delta$.

Lemma 1.18. *Let* δ *be as above and set* $k = n-2w-p+4+\delta$. *Then* $[G_i,G_j] \leq G_{i+j+k}$ *for all* $i,j \geq 1$.

Proof. This follows from Lemma 1.14 since

$$[s_i,s_{i+t}] \equiv s_x^{e(i,t,\delta)} \bmod G_{x+1}$$

where $x = m(i,t)+\delta = 2i+t+n-2w-p+4+\delta = i+(i+t)+k$.

The assumption in Theorem 4 that $w \leq (n-2p+2\delta+12)/4$ is derived from Lemma 1.18 and the condition that G_1 be of class 2. For, by Lemma 1.18, $[G_1,G_1] = [G_1,G_2] \leq G_{3+k}$. So

$$[G_1, G_1, G_1] \leq [G_{3+k}, G_1] \leq G_{4+2k},$$

and if we take $4 + 2k \geq n$, i.e. $n - 4w - 2p + 2\delta + 12 \geq 0$, we have $[G_1, G_1, G_1] = 1$.

The bound $n - 4w - 2p + 2\delta + 12 \geq 0$ is a sufficient condition for G_1 to be of class 2; it may not be necessary. Consider the following case where $n - 4w - 2p + 2\delta + 12 = -1$. Calculating $[s_1, s_2, s_1]$ by Lemma 1.15 we get

$$[s_1, s_2, s_1] = s_{n-1}^{-\alpha}$$

where

$$\alpha = a(1, N(1) + \delta) \sum_{j=0}^{w-i} a(i + j, N(1 + j) + \delta)(-1)^j \binom{n - 2w - p + 5 + \delta - j}{j}.$$

If G_1 is to be of class 2 we must have $\alpha \equiv 0 \bmod p$. But, by the above, α may or may not be divisible by p; it depends on the values assumed by the $a(i, N(i) + j)$. In short if w is large enough so that $n - 4w - 2p + 2\delta + 12 < 0$ the question of the existence of the groups in question depends on the particular values of the $a(i, N(i) + j)$.

Finally, it is easy to see that if G_1 is to be of class 2 and G_w is the first abelian member of the lower central series for G then $w \leq (n - p + 7 + \delta)/3$. For under these circumstances $[G_1, G_2]$ must be abelian, and, consequently, is contained in G_w. But then, by Lemma 1.17,

$$[s_1, s_2] = \prod_{r \geq \delta} s_{n-2w-p+7+\delta}^{e(1,1,r)} \in G_w,$$

so $n - 2w - p + 7 + \delta \geq w$.

2. In this section we introduce two products which play a central role in the extension problem.

The products are defined for $i \geq 1$ by

$$\beta(i, 1) = \prod_{a=1}^{p-1} \prod_{u=a}^{p-1} \lambda(i, a, u - a)^{-\binom{p}{u+1}}$$

and

$$\beta(i, 0) = \prod_{a=2}^{p-1} \prod_{u=a}^{p-1} \lambda(i, a - 1, u - a)^{-\binom{p}{u+1}}.$$

The commutator $\lambda(i, t, v)$ is defined as in Lemma 1.1. If we apply Lemma 1.2 we have

Lemma 2.1. *Let $\beta(i, 0)$ and $\beta(i, 1)$ be defined as above. Then*

$$\beta(i, 1) = \prod_{j=0}^{p-2} \prod_{\ell=0}^{p-2-j} \tau(i + j, \ell)^{m(j,\ell)}$$

where

$$m(j,\ell) = (-1)^{j+1}\binom{\ell+j+1}{j+1}\binom{p+j}{\ell+2j+2},$$

$$\beta(i,0) = \prod_{j=0}^{p-3}\prod_{\ell=0}^{p-3-j}\tau(i+j,\ell)^{n(j,\ell)},$$

where $n(j,\ell) = (-1)^{j+1}\binom{\ell+j+1}{j+1}\binom{p+j}{\ell+2j+3}$.

Proof. Applying Lemma 1.2 we get a four-fold product with a, u, j, and ω as indices of multiplication. If we hold j, ω, and u and let a run through its range we find that

$$\beta(i,1) = \prod_{j=0}^{w-1-i}\prod_{\omega=0}^{j}\prod_{u=1}^{p-1}\tau(i+j,u-j-1-\omega)^{m(j,\omega,u)}$$

where $m(j,\omega,u) = (-1)^{j+1}\binom{j}{\omega}\binom{p}{u+1}\binom{u-\omega}{j+1}$.

Next, holding j fixed and gathering together those terms where $u-j-1-\omega = x$, we get

$$\beta(i,1) = \prod_{j=0}^{w-1-i}\prod_{x=0}^{p-2}\tau(i+j,x)^{m(j,x)}$$

where $m(j,x) = (-1)^{j+1}\binom{x+j+1}{j+1}\binom{p+j}{x+2j+2}$. This completes the proof of the first part of Lemma 2.1. The proof of the second part is similar.

Later on we shall see that we must have $\beta(i,1) = 1$. Thus we will have two representations of 1:

$$B(i) = \prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j}\tau(i+j,\ell)^{k(j,\ell)} = 1$$

where $k(j,\ell) = (-1)^{j}\binom{\ell+j}{j}\binom{p+j}{\ell+2j+2}$, and

$$\beta(i,1) = \prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j}\tau(i+j,\ell)^{m(j,\ell)}$$

where $m(j,\ell) = (-1)^{j+1}\binom{\ell+j+1}{j+1}\binom{p+j}{\ell+2j+2}$. If we know G is a group we have two equations and no problems. However if we are trying to construct G the question of the consistency of these two equations arises.

We shall also have to have $\beta(i,0) = 1$ for $i \geq 2$. This condition is slightly different from the two above in that it imposes a defining congruence on the parameter $a(i,n-p+2)$. To see this one must prove:

Lemma 2.2. *Let* $\beta(i,0)$ *be defined as above,* $a(i,N(i)+q)$ *as in Lemma* 1.8, *and* $E(j)$ *as in Lemma* 1.14. *Set* $A(i,q) = a(i,N(i)+q)$ *and* $E(j) = 0$ *for* $j < 0$. *Let*

$$\psi(j,\ell) = \sum_{\nu=0}^{p-4-2j} E(\ell - 2j - \nu)\frac{(-1)^{j+1}}{p}\binom{\nu+j+1}{j+1}\binom{p+j}{\nu+2j+3},$$

$$S(i,r) = \sum_{j=0}^{p-3}\sum_{\ell=0}^{r} A(i+j,r-\ell)(-1)^{j+1}\binom{p-2-j+\ell}{j+1}\binom{p+j}{p+\ell},$$

$$T(i,r) = \sum_{j=0}^{\rho}\sum_{\ell=2j}^{r} A(i+j,r-\ell)\psi(j,\ell)$$

where $\rho = (p-5)/2$ *and* $W(i,r) = S(i,r+2) + T(i,r)$. *Then*

$$\beta(i,0) = s_{n(i)+2}^{S(i,0)} s_{n(i)+3}^{S(i,1)} \prod_{r=0}^{2(w-i)-5} s_{n(i)+4+r}^{W(i,r)}$$

where $n(i) = n - 2(w-i)$.

The proof is similar to the proof of Lemma 1.15. Furthermore an argument similar to that given for Lemma 1.16 shows that the condition $\beta(i,0) = 1$ is equivalent to a system of congruences which can be considered as defining congruences for the parameters $A(i,r+2) = a(i,N(i)+r+2)$. Since $N(i)+r+2 \leq n-p+2$ we have a set of congruences in which

$$a(i,N(i)),\ldots,a(i,n-p+1),a(i,n-p+2); \qquad i = w-2,\ldots,2,$$

can be considered as dependent variables.

There is a relation between the three products under consideration:

Lemma 2.3. *Let* $B(i)$ *be defined as in Lemma* 1.6 *and* $\beta(i,1)$ *and* $\beta(i,0)$ *as in Lemma* 2.1. *Then*

$$[\beta(i,0),s]\beta(i,1)^{-1} = B(i).$$

Proof. We have

$$[\beta(i,0),s] = \prod_{j=0}^{p-3}\prod_{\ell=1}^{p-2-j} \tau(i+j,\ell)^{n(j,\ell-1)},$$

$$\beta(i,1)^{-1} = \prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j} \tau(i+j,\ell)^{-m(j,\ell)}.$$

In addition

$$n(j,\ell-1) - m(j,\ell) = (-1)^{j+1}\binom{\ell+j}{j+1}\binom{p+j}{\ell+2j+2} + (-1)^{j}\binom{\ell+j+1}{j+1}\binom{p+2}{\ell+2j+2} = (-1)^{j}\binom{\ell+j}{j}\binom{p+j}{\ell+2j+2} = k(j,\ell).$$

Lemma 2.3 follows from this.

To return to the consistency question: Lemma 2.3 shows that if $B(i) = 1$ and $\beta(i,1) = 1$ then $[\beta(i,0),s] = 1$. But if

$$[\beta(i,0),s] = s_{n(i)+3}^{S(i,0)} s_{n(i)+4}^{S(i,1)} \prod_{r=0}^{2(w-i)-6} s_{n(i)+5+r}^{W(i,r)} = 1$$

then the congruences we get for $a(i,N(i)),\ldots,a(i,n-p+1)$ from $\beta(i,0) = 1$ are equivalent to those derived from $B(i) = 1$. Furthermore since $[\beta(i,0),s] = 1$ we can get $\beta(i,0) = 1$ by choosing the parameter $a(i,n-p+2)$ suitably.

In sum we shall take $\beta(i,0) = 1$, $i \geq 2$, and $B(1) = 1$ as defining relations for G in Theorem 2.

The next result is a very useful reduction formula for pth powers of commutators.

Lemma 2.4. *Suppose that* $B(i) = \beta(i+1,0) = 1$. *Then*

$$[s_i, s_{i+1}]^p = \prod_{t=1}^{p-2} [s_{i+1}, s_{i+1+t}]^{\binom{p}{t+2}}.$$

Proof. By Lemma 1.1 and induction on u,

$$\lambda(i,u,0) = \tau(i,u-1) \prod_{a=2}^{u} \lambda(i+1,a-1,u-a)^{-1}\lambda(i+1,a-1,u-1-a)^{-1}.$$

By Lemma 1.6,

$$B(i) = \prod_{u=1}^{p-1} \lambda(i,u,0)^{\binom{p}{u+1}}.$$

Thus $B(i) = P_1 P_2 P_3$ where

$$P_1 = \prod_{u=1}^{p-1} \tau(i,u-1)^{\binom{p}{u+1}},$$

$$P_2 = \prod_{u=2}^{p-1} \prod_{a=2}^{u} \lambda(i+1,a-1,u-a)^{-\binom{p}{u+1}},$$

$$P_3 = \prod_{u=2}^{p-1} \prod_{a=2}^{u-1} \lambda(i+1,a-1,u-1-a)^{-\binom{p}{u+1}}.$$

Now if $g \in G_i$ where G_i is abelian we have, from Lemma 1.3,

$$g^p[g,s]^{\binom{p}{2}} \cdots [g,s,\underset{p-1}{\ldots},s]^{\binom{p}{p}} = 1.$$

Consequently,

$$[P_1,s] = \prod_{u=1}^{p-1} \tau(i,u)^{\binom{p}{u+1}} = \tau(i,0)^{-p}.$$

Next, by the definition of $\beta(i+1,0)$, $P_2 = \beta(i+1,0)$. Finally,

$$\begin{aligned}
[P_3, s] &= \prod_{u=2}^{p-1} \prod_{a=2}^{u-1} \lambda(i+1, a-1, u-a)^{-\binom{p}{u+1}} \\
&= \prod_{a=2}^{p-1} \prod_{u=a}^{p-1} \lambda(i+1, a-1, u-a)^{-\binom{p}{u+1}} \otimes \prod_{a=2}^{p-2} \lambda(i+1, a-1, 0)^{\binom{p}{a+1}} \\
&= \beta(i+1,0) \prod_{t=1}^{p-2} \lambda(i+1, t, 0)^{\binom{p}{t+2}}.
\end{aligned}$$

Thus

$$[B(i), s] = \tau(i,0)^{-p}[\beta(i+1,0), s]\beta(i+1,0) \prod_{t=1}^{p-2} \lambda(i+1, t, 0)^{\binom{p}{t+2}}.$$

Lemma 2.4 is a consequence of this equation.

If we apply Lemma 1.2 to Lemma 2.4 we get

Lemma 2.5. *Suppose that* $B(i) = \beta(i+1,0) = 1$. *Then*

$$\tau(i,0)^p = \prod_{j=1}^{p-2} \prod_{\ell=0}^{p-2-j} \tau(i+j, \ell)^{f(j,\ell)}$$

where $f(j,\ell) = (-1)^{j-1}\binom{\ell+j-1}{j-1}\binom{p+j-1}{\ell+2j+1}$.

The next few results deal with reducing the products in Lemmas 1.3 and 1.5 to commutators of the form $\tau(i,\nu)$ and the functions $\beta(i,0)$, $\beta(i,1)$. To begin we need several binomial identities.

Lemma 2.6.

$$\binom{k+2t-\ell-1}{t-1}\binom{k+t}{k+2t-\ell} = \sum_{\nu=1}^{\ell+1-t} \binom{\ell-\nu}{t-1}\binom{k+t-\nu}{\ell-\nu},$$

$$\sum_{j=0}^{n} (-1)^j \binom{a+j}{a}\binom{b}{n-j} = \binom{b-a-1}{n}.$$

Proof. Let L denote the left-hand side of the first equation. It is easy to check that for $a \geq 1$

$$L = \sum_{\nu=0}^{a} \binom{\ell-\nu}{t-1}\binom{k+t-a}{\ell-a} + \binom{k+2t-\ell-1}{t-1}\binom{k+t-a}{k+2t-\ell}.$$

The first equation is a consequence of this relation.

The second equation can be proved by equating coefficients in the two expansions of $(1+x)^{-a-1}(1+x)^b$.

Lemma 2.7. *Let* $q(i)$ *be defined as in Lemma* 1.3. *Set*

$$U_1(j,\ell) = \sum_{r=1}^{j+1}\binom{2r-1}{r}\binom{r+j+1}{2r}\binom{p}{j+\ell+2+r},$$

$$U_2(j,\ell) = \sum_{r=0}^{j+1}(-1)^j\binom{j+1}{r}\binom{\ell+j+1}{\ell}\binom{p}{j+\ell+2+r},$$

and $U(j,\ell) = U_1(j,\ell) + U_2(j,\ell)$. *Then*

$$q(i) = \prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j}\tau(i+j,\ell)^{-U(j,\ell)}.$$

Proof. According to Lemma 1.3,

$$q^{-1}(i) = \prod_{\ell=0}^{p-2}\prod_{k=\ell+1}^{p-1}[s_{i+\ell}, s_{i+k}]^{B(p,k,\ell)}$$

where $B(p,k,\ell) = \sum_{u=0}^{\ell+1}[b_1(k,\ell,u) + b_2(k,\ell,u)]\binom{p}{k+1+u}$ with $b_1(k,\ell,u) = \binom{\ell+1}{u}\binom{k+u}{\ell+1}$ and, by Lemma 2.7,

$$b_2(k,\ell,u) = \sum_{v=1}^{\ell+1-u}\binom{\ell-v}{u-1}\binom{k+u-v}{\ell-v}.$$

Set $k = \ell + t$, $t = 1,\dots,p-1-\ell$ in the above product for $q^{-1}(i)$. Then, applying Lemma 1.2 and gathering similar terms together, we have

$$q^{-1}(i) = \prod_{x=0}^{w-1-i}\prod_{y=0}^{p-2}\tau(i+x,y)^{U(x,y)}$$

where

$$U(x,y) = \sum_{j=0}^{x}\sum_{\omega=0}^{j}(-1)^j\binom{j}{\omega}\binom{y+j}{j}B(p,x+y+\omega+1,x-j).$$

So, in view of the equation for $B(p,k,\ell)$ in the preceding paragraph, let

$$U_i^* = \sum_{j=0}^{x}\sum_{\omega=0}^{j}\sum_{u=0}^{x-j+1}(-1)^j\binom{j}{\omega}\binom{y+j}{y}$$
$$\cdot\, b_i(x+y+\omega+1,x-j,u)\binom{p}{x+y+\omega+2+u}$$

for $i = 1, 2$. Then $U(x,y) = U_1^* + U_2^*$.

To evaluate U_1^*, fix j and gather together those terms where $\omega + u = r$ to get

$$U_1^* = \sum_{j=0}^{x}\sum_{r=0}^{x+1}(-1)^j\binom{y+j}{j}\binom{x+y+r+1}{x+1-j}$$
$$\cdot\binom{p}{x+y+r+2}\binom{x+1}{r}.$$

Then apply the second identity of Lemma 2.7 to get

$$U_1^* = \sum_{r=0}^{x} \binom{x+1}{r}\left[\binom{x+r}{x+1} + (-1)^x\binom{x+y+1}{y}\right]\binom{p}{x+y+r+2}.$$

The evaluation of U_2^* is similar. One gets

$$U_2^* = \sum_{r=1}^{x} \binom{2r-1}{r}\binom{x+r}{2r}\binom{p}{x+y+r+2}.$$

Finally since $\binom{x+1}{r}\binom{x+r}{x+1} + \binom{2r-1}{r}\binom{x+r}{2r} = \binom{2r-1}{r}\binom{x+r+1}{2r}$, U_2^* and the first part of U_1^* combine, in the notation of Lemma 2.8, as $U_1(x,y)$. If we label the remaining part of U_1^* as $U_2(x,y)$, we have Lemma 2.7.

Lemma 2.8. *Let $U_1(j,\ell)$ be defined as in Lemma 2.7. Let $\beta(i,0)$ and $\beta(i,1)$ be as in Lemma 2.1. Then*

$$q(i) = \beta(i,0)\beta(i,1)\prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j}\tau(i+j,\ell)^{-U_1(j,\ell)}.$$

Proof. According to Lemma 2.8,

$$\begin{aligned}-U_2(j,\ell) &= (-1)^{j+1}\binom{\ell+j+1}{j+1}\sum_{r=0}^{j+1}\binom{j+1}{r}\binom{p}{p-j-\ell-r-2} \\ &= (-1)^{j+1}\binom{\ell+j+1}{j+1}\binom{p+j+1}{\ell+2j+3}.\end{aligned}$$

But this is the exponent we get for $\tau(i+j,\ell)$ when we form the product $\beta(i,0)\beta(i,1)$.

This, for the moment, ends the discussion of the product $q(i)$. The next few results are about the product that appears in Lemma 1.5. To begin we have:

Lemma 2.9. *Let $q^{-1}(i)r(i)$ be defined as in Lemma 1.5. Set*

$$\begin{aligned}v(x,y,r) = \binom{2r}{r}\binom{x+1+r}{2r} &+ (-1)^{r+1}\binom{y+r}{y}\binom{p-r-y-1}{x+1-r} \\ &+ (-1)^x\binom{x+1}{r}\binom{x+y+1}{y}\end{aligned}$$

and

$$V(x,y) = \sum_{r=0}^{x+1} v(x,y,r)\binom{p}{x+y+r+2}.$$

Then

$$q^{-1}(i)r(i) = \prod_{x=0}^{p-2}\prod_{y=0}^{p-2-x}\tau(i+x,y)^{V(x,y)}.$$

Proof. Applying Lemma 1.2 to Lemma 1.5 we get

$$q^{-1}(i)r(i) = \prod_{x=0}^{w-1-i} \prod_{y=0}^{p-2} \tau(i+x,y)^{V(x,y)}$$

where

$$V(x,y) = \sum_{j=0}^{x} \sum_{\omega=0}^{j} (-1)^j \binom{j}{\omega}\binom{y+j}{y} D(p,x-j,x+y+1+\omega),$$

$$D(p,x-j,x+y+1+\omega) = -\binom{p}{x-j+1}\binom{p}{x+y+2+\omega} + D^*,$$

and

$$D^* = \sum_{u=0}^{x-j+1} \left[2\binom{x-j+1}{u}\binom{x+y+1+\omega+u}{x-j+1} + \binom{x-j}{u}\binom{x+y+1+\omega+u}{x-j}\right]\binom{p}{x+y+2+\omega+u}.$$

Thus $V(x,y) = -V_1(x,y) + 2V_2(x,y) + V_3(x,y)$ where

$$\begin{aligned} V_1(x,y) &= \sum_{j=0}^{x} \sum_{\omega=0}^{j} (-1)^j \binom{j}{\omega}\binom{y+j}{j}\binom{p}{x-j+1}\binom{p}{x+y+2+\omega} \\ &= \sum_{r=0}^{x+1} (-1)^r \binom{y+r}{y}\binom{p-r-1-y}{x+1-r}\binom{p}{x+y+2+r} \\ &\quad - \sum_{r=0}^{x+1} (-1)^{x+1}\binom{x+1}{r}\binom{y+x+1}{y}\binom{p}{x+y+2+r}. \end{aligned}$$

(To get this extend the range of summation to $j = x + 1$ in the original sum, reverse the order of summation, and then apply Lemma 2.6.)

$$V_2(x,y) = \sum_{r=0}^{x+1} \binom{x+1}{r}\left[\binom{x+r}{x+1} + (-1)^x\binom{x+y+1}{y}\right]\binom{p}{x+y+2+r}.$$

(One starts with the first part of D^* and then argues as in the proof of Lemma 2.8.)

$$V_3(x,y) = \sum_{r=0}^{x} \binom{x}{r}\binom{x+r}{r}\binom{p}{x+y+2+r}.$$

Combining these sums, one gets the $V(x,y)$ of Lemma 2.10.

The quantity $V(x,y)$ of Lemma 2.9 can be simplified. To this end we have

Lemma 2.10. *Let*

$$A(x,y) = \sum_{r=0}^{x+1} \binom{2r}{r}\binom{x+1+r}{2r}\binom{p}{x+y+r+2}$$

and

$$B(x,y) = \sum_{r=0}^{x+1} (-1)^r \binom{y+r}{y} \binom{p-r-y-1}{x+1-r} \binom{p}{x+y+r+2}.$$

Then $A(x,y) = B(x,y)$.

Proof. If we set $y = p - x - 2 - t$ we have

$$B(x,y) = \sum_{r=0}^{x+1} (-1)^r \binom{p}{t-r} \binom{x+1+t-r}{x+1-r} \binom{p-x-2-t+r}{r}.$$

By the second part of Lemma 2.7,

$$\binom{p-x-2-t+r}{r} = \sum_{j=0}^{r} (-1)^j \binom{x+1+j}{x+1} \binom{p-t+r}{r-j}.$$

Substituting this in the first equation above and rearranging we get

$$B(x,y) = \sum_{j=0}^{t} \binom{p}{t-j} \binom{x+1+j}{x+1} \sum_{r=j}^{t} (-1)^{r+j} \binom{t-j}{t-r} \binom{x+1+t-r}{x+1-r}.$$

Making the substitution $y = p - x - 2 - t$ in the sum for $A(x,y)$ we get

$$A(x,y) = \sum_{j=0}^{t} \binom{2j}{j} \binom{x+1+j}{2j} \binom{p}{t-j}.$$

Comparing coefficients and cancelling common factors we see that the equation $B(x,y) = A(x,y)$ is equivalent to the equation

$$\sum_{r=j}^{t} (-1)^{r+j} \binom{t-j}{t-r} \binom{x+1+t-r}{t} = \binom{x+1}{j}.$$

But this identity is a special case of the identity

$$\sum_{k=0}^{N} (-1)^k \binom{A}{k} \binom{B-1+N-k}{B-1} = (-1)^N \binom{A-B}{N},$$

which is derived by equating coefficients in the expansion of $(1+x)^A(1+x)^{-B}$. This completes the proof of Lemma 2.11.

If we combine Lemmas 2.10 and 2.11 we have

$$\begin{aligned} V(x,y) &= (-1)^x \binom{y+x+1}{x+1} \sum_{r=0}^{x+1} \binom{x+1}{r} \binom{p}{x+y+r+2} \\ &= (-1)^x \binom{y+x+1}{x+1} \binom{p+x+1}{y+2x+3}. \end{aligned}$$

But this, except for sign, is the exponent of $\tau(i+x,y)$ in the product $\beta(i,0)\beta(i,1)$. This proves

Lemma 2.11. *Let* $q^{-1}(i)r(i)$ *be defined as in Lemma* 1.5; $\beta(i,0)$ *and* $\beta(i,1)$ *be as in Lemma* 2.1. *Then*

$$q^{-1}(i)r(i) = (\beta(i,0)\beta(i,1))^{-1}.$$

Later we shall need another identity similar to the one given in Lemma 2.10:

Lemma 2.12. *Let*

$$M_1(j,\ell) = \sum_{r=0}^{j} (-1)^r \binom{\ell+r}{r}\binom{p}{j-r}\binom{p+r}{\ell+j+2+r}$$

and

$$M_2(j,\ell) = \sum_{r=1}^{j+1} \binom{2r-1}{r}\binom{j+r}{j+1-r}\binom{p}{\ell+j+r+1}.$$

Then $M_1(j,\ell) = M_2(j,\ell)$.

Proof. To start we have

$$\begin{aligned} M_1(j,\ell) &= \sum_{r=0}^{j} (-1)^r \binom{\ell+r}{r}\binom{p}{j-r} \sum_{a=0}^{r} \binom{r}{a}\binom{p}{\ell+j+2+a} \\ &= \sum_{a=0}^{j} \binom{p}{\ell+j+2+a} \sum_{r=a}^{j} (-1)^r \binom{\ell+a}{\ell}\binom{\ell+r}{\ell+a}\binom{p}{j-r} \\ &= \sum_{a=0}^{j} \binom{p}{\ell+j+2+a}(-1)^a \binom{\ell+a}{\ell}\binom{p-1-\ell-a}{j-a}. \end{aligned}$$

If we set $\ell = p-1-j-t$ in this last sum we have

$$\begin{aligned} M_1(j,\ell) &= \sum_{a=0}^{j} (-1)^a \binom{j+t-a}{j-a}\binom{p}{t-1-a}\binom{p-1-j-t+a}{a} \\ &= \sum_{a=0}^{t-1} (-1)^a \binom{j+t-a}{j-a}\binom{p}{t-1-a} \\ &\qquad \cdot \sum_{k=0}^{a} (-1)^k \binom{p-t+a+1}{a-k}\binom{j+1+k}{j+1} \\ &= \sum_{k=0}^{t-1} \binom{p}{t-1-k}\binom{j+1+k}{j+1} \sum_{a=k}^{t-1} (-1)^{a+k} \binom{j+t-a}{t}\binom{t-1-k}{t-1-a}. \end{aligned}$$

To continue, in the defining sum for $M_2(j,\ell)$, replace r by $k+1$ and y by $p-1-j-t$. Then

$$M_2(j,\ell) = \sum_{k=0}^{\ell-1} \binom{j+1+k}{j+1}\binom{j+1}{k+1}\binom{p}{\ell-1-k}$$

and the problem of showing that $M_1(j,\ell) = M_2(j,\ell)$ reduces to one of showing that $\sum_{a=k}^{\ell-1} (-1)^{k+a}\binom{j+\ell-a}{\ell}\binom{\ell-1-k}{\ell-1-a} = \binom{j+1}{k+1}$. But this can be established by equating coefficients in the different expansions of $(1+x)^A(1+x)^{-B}$.

We shall also need:

Lemma 2.13. *If $j \geq 0$ then*

$$\binom{\binom{p}{j+1}}{2} = \sum_{r=1}^{j+1} \binom{2r-1}{r}\binom{r+j+1}{j+1-r}\binom{p}{j+r+1}.$$

This is easy to prove. Merely note that the right-hand side of the equation is equal to $\frac{1}{2}\binom{p}{j+1}\sum_{r=1}^{j+1}\binom{p-j-1}{r}\binom{j+1}{j+1-r}$.

3. The main result of this section is the relation between $\beta(i,1)$ and the $B(i)$ stated in Lemma 3.11. The equation there is based on several identities (given in Lemmas 3.1 and 3.7) which we prove first.

Lemma 3.1. *Let ℓ and N be nonnegative integers. Let*

$$S_q = \sum_{k=0}^{N} \binom{k+\ell-q}{\ell-q}\binom{N-k+q}{q}\binom{p+k-1}{2k+2+\ell-q}\binom{p+N-k}{2N-2k+q+2}$$

and

$$T_q = \sum_{k=0}^{N} \binom{k+\ell-q}{\ell-q}\binom{N-k+q}{q}\binom{p+k-1}{2k+1+\ell-q}\binom{p+N-k}{2N-2k+q+3}.$$

Then

$$\sum_{q=0}^{\ell} (S_q - T_q) + \frac{p+N+1}{n+2}\binom{N+1+\ell}{\ell}\binom{p+N}{2N+3+\ell} = \binom{N+1+\ell}{\ell}\binom{p+N}{2N+3+\ell}p.$$

The proof of Lemma 3.1 to be given here is due to Karl Goldberg. It is a bit complicated, but it is considerably simpler than my original proof. A third proof (and generalization) of Lemma 3.1 was discovered by L. Carlitz; his argument is similar to Goldberg's.

To start we have:

Lemma 3.2. *Let*

$$F_{p,j}(x,y) = \sum_{q=0}^{\infty} \sum_{k=0}^{\infty} \binom{k+q}{q} \binom{p+k}{2k+j+q} x^q y^k.$$

Then

$$F_{p,2}(x,y)F_{p-1,2}(x,y) = \sum_{l=0}^{\infty} \sum_{N=0}^{\infty} \left(\sum_{q=0}^{l} S_q \right) x^l y^N$$

and

$$F_{p,3}(x,y)F_{p-1,1}(x,y) = \sum_{l=0}^{\infty} \sum_{N=0}^{\infty} \left(\sum_{q=0}^{l} T_q \right) x^l y^N.$$

This can be proved by multiplying out the quantities involved.

Lemma 3.3. *For* $j \geq 1$,

$$\sum_{p=0}^{\infty} F_{p,j}(x,y) z^p = \frac{z^j}{(1-z)^{j-1}[1-(x+y+2)z+(x+1)z^2]}.$$

Furthermore, letting $F_{p,j} \equiv F_{p,j}(x,y)$,

$$F_{p,j+1} = \sum_{r=0}^{p-1} F_{r,j}.$$

The first part of this lemma follows from the definition of Lemma 3.2, rearrangements, and several well-known identities. The second is an easy consequence of the first.

Lemma 3.4. *Let* α *and* β *be functions of* x *and* y *such that* $\alpha + \beta = x + y + 2$, $\alpha\beta = x + 1$. *Let* $g(p) = (\alpha^p - \beta^p)/(\alpha - \beta)$. *Then*

(1) $$g(p+1) = (x+y+2)g(p) - (x+1)g(p-1),$$

(2) $$(g(p))^2 = g(p+1)g(p-1) + (x+1)^{p-1}.$$

Furthermore

(3) $$F_{p,1} \equiv F_{p,1}(x,y) = g(p),$$

(4) $$yF_{p,2} = -1 - (x+1)g(p-1) + g(p),$$

(5) $$= -1 - (x+y+1)g(p) + g(p+1),$$

(6) $$y^2F_{p,3} = (x+y-py) + (x+1)(x+y)g(p-1) - xg(p).$$

Proof. Equations (1) and (2) are trivial. As for (3), we have

$$\sum_{p=0}^{\infty} F_{p,1}(x,y)z^p = \frac{z}{(1-\alpha z)(1-\beta z)} = \frac{1}{\alpha-\beta}\left(\frac{\alpha z}{1-\alpha z} - \frac{\beta z}{1-\beta z}\right).$$

Thus $F_{p,1} = (\alpha^p - \beta^p)/(\alpha - \beta) = g(p)$. Next, by the second part of Lemma 3.3,

$$F_{p,2} = \sum_{r=0}^{p-1} F_{p,1} = \frac{1}{\alpha-\beta}\left(\frac{1-\alpha^p}{1-\alpha} - \frac{1-\beta^p}{1-\beta}\right) = -\frac{1}{y} - \frac{1}{\alpha-\beta}\left(\frac{\alpha^p}{1-\alpha} - \frac{\beta^p}{1-\beta}\right)$$
$$= (1/y)(-1 - (x+1)g(p-1) + g(p)).$$

This gives us (4). Equation (5) follows from (4) and (1).

Finally,

$$F_{p,3} = \sum_{r=0}^{p-1} -\frac{1}{y} - \frac{1}{\alpha-\beta}\left(\frac{\alpha^r}{1-\alpha} - \frac{\beta^r}{1-\beta}\right)$$
$$= -\frac{p}{y} + \frac{(x+y)}{y^2} + \frac{1}{y^2}(g(p) - 2(x+1)g(p-1) + (x+1)^2 g(p-2)).$$

Applying (1) to the $g(p-2)$ term in this last expression we get (6).

Lemma 3.5. *We have*

$$y^2 F_{p,2} \cdot F_{p-1,2} = 1 + (2x+y+2)g(p-1) - 2g(p)$$
$$+ (x+1)(x+y)g^2(p-1) - xg(p)g(p-1) + (x+1)^{p-1}.$$

This follows from (4), (5) and (1) and (2) of Lemma 3.4.

Lemma 3.6. *Let* $S = F_{p,2}F_{p-1,2} - F_{p,3}F_{p-1,1}$. *Then*

$$S = (1/y^2)(1 + (x+1)^{p-1} + (x+py+2)F_{p-1,1} - 2F_{p,1}).$$

This follows from Lemma 3.5 and equations (3) and (6) of Lemma 3.4.

Now S of Lemma 3.6 is, by Lemma 3.2, a polynomial in x and y. In particular, the coefficients of $1/y$ and $1/y^2$ in the expression of Lemma 3.6 must be zero and we have

$$S = p\sum_{q=0}^{\infty} \binom{q+1}{1}\binom{p}{q+3} x^q$$
$$+ (x+py+2)\sum_{q=0}^{\infty}\sum_{k=2}^{\infty} \binom{k+q}{q}\binom{p+k-1}{2k+q+1} x^q y^{k-2}$$
$$- 2\sum_{q=0}^{\infty}\sum_{k=2}^{\infty} \binom{k+q}{q}\binom{p+k}{2k+q+1} x^2 y^{k-2}.$$

Reindexing we find that

$$S = \sum_{\ell=1}^{\infty} \sum_{N=0}^{\infty} \binom{N+\ell+1}{\ell-1}\binom{p+n+1}{2N+\ell+4} x^{\ell} y^{N}$$

$$+ \sum_{\ell=0}^{\infty} \sum_{N=0}^{\infty} p\binom{N+\ell+1}{\ell}\binom{p+n}{2N+\ell+3} x^{\ell} y^{N}$$

$$+ \sum_{\ell=0}^{\infty} \sum_{N=0}^{\infty} 2\binom{N+\ell+2}{\ell}\binom{p+N+1}{2N+\ell+5} x^{\ell} y^{N}$$

$$+ \sum_{\ell=0}^{\infty} \sum_{N=0}^{\infty} (-2)\binom{N+\ell+2}{\ell}\binom{p+N+2}{2N+\ell+5} x^{\ell} y^{N}.$$

This equation and Lemma 3.2 yield

$$\sum_{q=0}^{\ell} (S_q - T_q) = p\binom{N+\ell+1}{\ell}\binom{p+N}{2N+\ell+3}$$
$$- \frac{p+N+1}{N+2}\binom{N+\ell+1}{\ell}\binom{p+N}{2N+\ell+3},$$

which is the conclusion of Lemma 3.1.

Lemma 3.7. *Let*

$$E_k(j,\ell) = \sum_{q=0}^{\ell} (-1)^{j-1}\binom{\ell-q+k}{k}\binom{p+k-1}{\ell-q+2k+1}$$
$$\cdot \binom{q+j-1-k}{j-1-k}\binom{p+j-1-k}{q+2(j-k)+1}$$

and

$$F_k(j,\ell) = \sum_{q=0}^{\ell} (-1)^{j}\binom{\ell-q+k-1}{k-1}\binom{p+k-2}{\ell-q+2k}$$
$$\cdot \binom{q+j-k}{j-k}\binom{p+j-k}{q+2(j-k)+2}.$$

Let

$$A(j,\ell) = \sum_{k=j-p+2}^{p-2} E_k(j,\ell), \qquad B(j,\ell) = \sum_{k=j-p+2}^{p-2} F_k(j,\ell),$$

and $S(j,\ell) = A(j,\ell) + B(j,\ell)$. *Then for* $p-1 \leq j \leq 2p-4$ *and* $0 \leq \ell \leq 2p - 4 - j$ *we have* $S(j,\ell) = 0$.

There are several steps to the proof of Lemma 3.7. The initial one consists of some rearrangements:

Lemma 3.8. *Let $J = j - p$ and $I = 2p - 4 - j - \ell$. Let*

$$E(q) = (-1)^{j-1} \sum_{t=0}^{I} \binom{\ell + j - p + 1 + t}{q} \binom{p - 2 - t}{\ell - q} \cdot \binom{j + \ell - q + t + 1}{2J + 2\ell - q + 5 + 2t} \binom{2p - 3 - \ell + q - t}{2p - 3 - \ell + q - 2t},$$

$$F(q) = (-1)^{j} \sum_{t=0}^{I} \binom{\ell + j - p + 1 + t}{\ell - q} \binom{p - 2 - t}{q} \cdot \binom{j + q + t}{2J + \ell + q + 4 + 2t} \binom{2p - 2 - q - t}{2p - 2 - q - 2t}.$$

Let $S(j, \ell)$ be defined as in Lemma 3.7. *Then*

$$S(j, \ell) = \sum_{q=0}^{\ell} (E(q) + F(q)).$$

Proof. To begin, let $T_k(q)$ denote a typical summand of $E_k(j, \ell)$. That is

$$T_k(q) = (-1)^{j-1} \binom{\ell - q + k}{k} \binom{p + k - 1}{\ell - q + 2k + 1} \cdot \binom{q + j - 1 - k}{j - 1 - k} \binom{p + j - 1 - k}{q + 2(j - k) + 1}.$$

Since $T_k(q) = 0$ when $k > p - 2 - \ell + q$, for the second binomial factor above is then zero, we have

$$A(j, \ell) = \sum_{k=j-p-2}^{p-2} \sum_{q=0}^{\ell} T_k(q) = \sum_{q=0}^{\ell} \sum_{k=j-p+2}^{p-2-q+\ell} T_k(q).$$

Now if we make the change of variable $k = j - p + 2 + t$, $t = 0, \ldots, 2p - 4 - j - \ell + q$ in the inner sum of the last double sum we find that this inner sum is equal to

$$\sum_{t=0}^{I+q} \binom{\ell - q + j - p + t + 2}{j - p + t + 2} \binom{q + p - 3 - t}{p - 3 - t} \cdot \binom{j + t + 1}{2(j - p) + \ell - q + 5 + 2t} \binom{2p - 3 - t}{2p - 3 + q - 2t}$$

where $I = 2p - 4 - j - \ell$. If we make the change of variable $t' = I + q - t$, $t = q, \ldots, I + q$ in this sum we get the quantity $E(q)$ of Lemma 3.8.

Similarly

$$\sum_{k=j-p+2}^{p-2} F_k(j, \ell) = \sum_{q=0}^{\ell} F(q).$$

Lemma 3.9. *Let $I = 2p-4-j-\ell$, $J = j-p$,*

$$E(1,q) = (-1)^{j-1}\sum_{t=0}^{I-1}\binom{\ell+j-p+1+t}{\ell-q}\binom{p-2-t}{q} \cdot \binom{j+q+t}{2J+\ell+q+5+2t}\binom{2p-3-q-t}{2p-3-q-2t}$$

and

$$F(1,q) = (-1)^{j}\sum_{t=0}^{I-1}\binom{\ell+j-p+2+t}{\ell-q}\binom{p-3-t}{q} \cdot \binom{j+q+t+1}{2J+\ell+q+6+2t}\binom{2p-4-q-t}{2p-4-q-2t}.$$

Then, for $q = 0, 1, \ldots, \ell$, $F(q) + E(\ell-q) = F(1,q) + E(1,q)$ and

$$S(j,\ell) = \sum_{q=0}^{\ell}(F(1,q) + E(1,q)).$$

Proof. Applying the identity

$$\binom{A}{B}\binom{C+1}{D+1} - \binom{A+1}{B+1}\binom{C}{D} = \binom{A}{B}\binom{C}{D+1} - \binom{A}{B+1}\binom{C}{D}$$

to the sum $F(q) + E(\ell-q)$ of Lemma 3.8 we get

$$F(q) + E(\ell-q) = (-1)^{j}\sum_{t=0}^{I}\binom{\ell+j-p+1+t}{\ell-q}\binom{p-2-t}{q}(\alpha(t)-\beta(t))$$

where

$$\alpha(t) = \binom{j+q+t}{2J+\ell+q+4+2t}\binom{2p-3-q-t}{2p-2-q-2t},$$

$$\beta(t) = \binom{j+q+t}{2J+\ell+q+5+2t}\binom{2p-3-q-t}{2p-3-q-2t}.$$

Now $\alpha(0) = 0$ and the corresponding sum is $F(1,q)$; $\beta(I) = 0$ and the corresponding sum is $E(1,q)$.

Lemma 3.10. *Let $I = 2p-4-j-\ell$, $J = j-p$, and i be a positive integer. Let for $q = 0, 1, \ldots, \ell$*

$$E(i,q) = (-1)^{j+i} \sum_{t=0}^{I-i} \binom{\ell + J + t + 1}{\ell - q}\binom{p - 1 - i - t}{q} \cdot \binom{j + t + q}{2J + \ell + q + 4 + i + 2t}\binom{2p - 2 - i - q - t}{2p - 2 - i - q - 2t},$$

$$F(i,q) = (-1)^{j+i-1} \sum_{t=0}^{I-i} \binom{\ell + J + t + 2}{\ell - q}\binom{p - 2 - i - t}{q} \cdot \binom{j + t + q + 1}{2J + \ell + q + 5 + i + 2t}\binom{2p - 3 - i - q - t}{2p - 3 - i - q - 2t},$$

and

$$G(i,q) = (-1)^{j+1-i} \sum_{t=0}^{I-i} \binom{\ell + J + t + 1}{\ell - q}\binom{p - 2 - i - t}{q - 1} \cdot \binom{j + t + q}{2J + \ell + q + 4 + i + 2t}\binom{2p - 2 - i - q - t}{2p - 2 - i - q - 2t}.$$

Then, for $q = 1, 2, \ldots, \ell + 1,$

$$\sum_{\nu=0}^{q-1} (E(i,\nu) + F(i,\nu)) = \sum_{\nu=0}^{q-1} (E(i+1,\nu) + F(i+1,\nu)) + G(i,q)$$

where $G(i, \ell + 1) = 0$.

We shall prove Lemma 3.10, for fixed i, by induction on q. That is, we have to show that $E(i,q) + F(i,q) + G(i,q) = E(i+1,q) + F(i+1,q) + G(i,q+1)$.

Now if $q \leq \ell$ we have

$$E(i,q) + G(i,q) + F(i,q) = \sum_{t=0}^{I-i} (-1)^{j+i}\binom{\ell + J + t + 1}{\ell - q}\binom{p - 2 - i - t}{q} \cdot [\alpha'(t) - \beta'(t)] + G(i,q+1)$$

where $\alpha'(t) = \binom{j+t+q}{2J+\ell+q+4+i+2t}\binom{2p-2-i-q-t}{2p-2-i-q-2t}$ and $\beta'(t) = \binom{j+t+q+1}{2J+\ell+q+5+i+2t}\binom{2p-3-i-q-t}{2p-3-i-q-2t}$. Applying the identity mentioned in the proof of Lemma 3.9 we have $\alpha'(t) - \beta'(t) = \alpha(t) - \beta(t)$ where

$$\alpha(t) = \binom{j + t + q}{2J + \ell + q + 4 + i + 2t}\binom{2p - 3 - i - q - t}{2p - 2 - i - q - 2t}$$

and

$$\beta(t) = \binom{j + t + q}{2J + \ell + q + 5 + i + 2t}\binom{2p - 3 - i - q - t}{2p - 3 - i - q - 2t}.$$

Note also that $\alpha(0) = 0$ and $\beta(I - i) = 0$. Thus, bringing these results together we have $E(i,q) + G(i,q) = E(i+1,q) + F(i+1,q) + G(i,q+1)$. This completes the proof of Lemma 3.10.

Lemma 3.7 follows from Lemma 3.10.

We now turn to the proof of

Lemma 3.11. *Let $B(i)$ be defined as in Lemma* 1.6 *and suppose that $B(j) = 1$ for $j \geq 1$. Let $\beta(i,0)$ and $\beta(i,1)$ be as in Lemma* 2.1 *and suppose that $\beta(j,0) = 1$ for $j \geq i+1$. Let*

$$D_k(l) = (-1)^k \binom{l+k-1}{k-1}\binom{p+k-1}{l+2k}.$$

Then

$$\beta(i,1) = B(i) \prod_{k=1}^{p-2} \prod_{l=0}^{p-2-k} [B(i+k), \overbrace{s,\ldots,s}^{l}]^{D_k(l)}.$$

That is, $\beta(i,1) = 1$.

Recall

$$\text{(I)} \qquad B(i) = \prod_{j=0}^{p-2} \prod_{l=0}^{p-2-j} \tau(i+j,l)^{k(j,l)}$$

where $k(j,l) = (-1)^j \binom{l+j}{j}\binom{p+j}{l+2j+2}$.

$$\text{(II)} \qquad \tau(i,0)^p = \prod_{j=1}^{p-2} \prod_{l=0}^{p-2-j} \tau(i+j,l)^{f(j,l)}$$

where $f(j,l) = (-1)^{j-1}\binom{l+j-1}{j-1}\binom{p+j-1}{l+2j+1}$. (The assumption $\beta(i+1,0) = 1$ of Lemma 3.11 is needed to insure that (II) holds; see Lemmas 2.4 and 2.5.)

We also have

$$\beta(i,1) = \prod_{j=0}^{p-2} \prod_{l=0}^{p-2-j} \tau(i+j,l)^{m(j,l)}$$

where $m(j,l) = (-1)^{j+1}\binom{l+j+1}{j+1}\binom{p+j}{l+2j+2}$.

Lemma 3.11 can be summarized as: If (I) and (II) hold then $\beta(i,1) = 1$.

We need to define several functions, sums, and products to carry out the proof of Lemma 3.11. First, let

$$H_j(l) = (-1)^j \binom{l+j}{j}\binom{p+j-1}{l+2j+1}, \qquad I(j,l) = \frac{p+j}{j+1} H_j(l).$$

Second, let $E_k(j,l)$ and $F_k(j,l)$ be defined as in Lemma 3.7. Finally, let

$$P(k) = \prod_{\nu=0}^{p-2-k} [B(i+k), \overbrace{s,\ldots,s}^{\nu}]^{D_k(\nu)},$$

$$Q_1(N) = \prod_{j=N+1}^{p-2} \prod_{l=0}^{p-2-j} \tau(i+j,l)^{I(j,l)},$$

$$Q_2(N) = \prod_{k=0}^{N-1} \prod_{j=N+1}^{p-2+k} \prod_{l=0}^{2p-4-j} \tau(i+j,l)^{E_k(j,l)},$$

$$Q_3(N) = \prod_{k=1}^{N} \prod_{j+N+1}^{p-2+k} \prod_{l=0}^{2p-4-j} \tau(i+j,l)^{F_k(j,l)}.$$

There are several steps to the proof of Lemma 3.11. The first is

Lemma 3.12. *Let $H_0(\nu)$ and $Q_1(0)$ be defined as above. Then*

$$B(i)\beta^{-1}(i,1)=\prod_{\nu=0}^{p-2}\tau(i,\nu)^{H_0(\nu)p}\otimes Q_1(0).$$

Proof. By (I) and the definition of $\beta(i,1)$,

$$B(i)\beta^{-1}(i,1)=\prod_{j=0}^{p-2}\prod_{\ell=0}^{p-2-j}\tau(i+j,\ell)^{e(j,\ell)}$$

where $e(j,\ell)=(-1)^j((p+j)/(j+1))\binom{\ell+j}{j}\binom{p+j-1}{\ell+2j+1}$. When $j=0$, $e(0,\ell)$ $=H_0(\ell)p$; when $j\geq 1$, $e(j,\ell)=I(j,\ell)$. Splitting the double product accordingly we get Lemma 3.2.

Lemma 3.13. *Let $E_0(j,\ell)$ be defined as above. Then*

$$\prod_{\nu=0}^{p-2}\tau(i,\nu)^{H_0(\nu)p}=\prod_{j=1}^{p-2}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{E_0(j,\ell)}.$$

Proof. By (II),

$$\prod_{\nu=0}^{p-2}\tau(i,\nu)^{H_0(\nu)p}=\prod_{\nu=0}^{p-2}\prod_{j=1}^{p-2}\prod_{q=0}^{p-2-j}\tau(i+j,q+\nu)^{H_0(\nu)f(j,q)}=\prod_{j=1}^{p-2}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{E_0(j,\ell)}$$

where $E_0(j,\ell)=\sum^{*}H_0(\nu)f(j,q)$ and the $*$ indicates the summation is over those ν and q when $0\leq\nu\leq p-2$, $0\leq q\leq p-2-j$, and $\nu+q=\ell$. Since $H_0(\nu)=\binom{p-1}{\nu+1}$ and $f(j,q)=(-1)^{j-1}\binom{q+j-1}{j-1}\binom{p+j-1}{q+2j+1}$ we have $H_0(\nu)=0$ for $\nu>p-2$ and $f(j,q)=0$ for $q>p-2-j$. Thus

$$E_0(j,\ell)=\sum_{q=0}^{\ell}(-1)^{j-1}\binom{q+j-1}{j-1}\binom{p+j-1}{q+2j+1}\binom{p-1}{\ell-q+1}.$$

Lemma 3.14. *Let $P(1)$ be defined as above. Then*

$$P(1)=\prod_{j=1}^{p-1}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{F_1(j,\ell)}.$$

Proof. By the definition of $P(1)$ and then (I),

$$\begin{aligned}P(1)&=\prod_{\nu=0}^{p-3}[B(i+1),\overbrace{s,\ldots,s}^{\nu}]^{D_1(\nu)}\\&=\prod_{\nu=0}^{p-3}\prod_{j=1}^{p-1}\prod_{q=0}^{p-1-j}\tau(i+j,\nu+q)^{D_1(\nu)k(j-1,q)}\\&=\prod_{j=1}^{p-1}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{F_1(j,\ell)}\end{aligned}$$

where

$$F_1(j,\ell) = \sum_{q=0}^{\ell} D_1(\ell - q)k(j-1,q)$$
$$= \sum_{q=0}^{\ell} (-1)^j \binom{q+j-1}{j-1}\binom{p+j-1}{q+2j}\binom{p-1}{\ell-q+2}.$$

Lemma 3.15. *We have*

$$B(i)\beta^{-1}(i,1)P(1) = \prod_{\ell=0}^{p-3} \tau(i+1,\ell)^{H_1(\ell)p} \otimes Q_1(1)Q_2(1)Q_3(1).$$

Proof. By Lemmas 3.2 and 3.3,

$$B(i)\beta^{-1}(i,1) = Q_1(0)\prod_{j=1}^{p-2}\prod_{\ell=0}^{2p-4-j} \tau(i+j,\ell)^{E_0(j,\ell)}$$

where $Q_1(0) = \prod_{j=1}^{p-2}\prod_{\ell=0}^{p-2-j} \tau(i+j,\ell)^{I(j,\ell)}$. By Lemma 3.14,

$$P(1) = \prod_{j=1}^{p-1}\prod_{\ell=0}^{2p-4-j} \tau(i+j,\ell)^{F_1(j,\ell)}.$$

Splitting off the part of these products corresponding to $j = 1$ we get the product $\prod_{\ell=0}^{2p-5} \tau(i+1,\ell)^{E(\ell)}$ where $E(\ell) = E_0(1,\ell) + F_1(1,\ell) + I(1,\ell)$. But $E(\ell) = H_1(\ell)p$ where $H_1(\ell)$ is defined as above; this is easy to see once the quantities involved are written out according to their definitions. Since $H_1(\ell) = 0$ for $\ell \geq p-2$ this gives us the product displayed in Lemma 3.15. The remnants of the double products give the $Q_i(1)$.

The methods of the last several lemmas lead to

Lemma 3.16. *Suppose that* $0 \leq N \leq p-2$. *Then*

$$B(i)\beta^{-1}(i,1)P(1)\cdots P(N) = \prod_{\nu=0}^{p-2-N} \tau(i+N,\nu)^{H_N(\nu)p} \otimes Q_1(N)Q_2(N)Q_3(N).$$

This is proved by induction on N, assuming that $N \leq p-3$. One employs (II) to reduce the displayed product, multiplies by $P(N+1)$, and then splits off the commutators $\tau(i+N+1,\ell)$ corresponding to $j = N+1$. The exponent of these commutators is

$$\sum_{k=0}^{N} (E_k(N+1,\ell) + F_{k+1}(N+1,\ell)) + I(N+1,\ell).$$

The connection between these quantities and those of Lemma 3.1 is

$$\sum_{k=0}^{N} F_{k+1}(N+1,\ell) = (-1)^{N+1}\sum_{q=0}^{\ell} S_q,$$

$$\sum_{k=0}^{N} E_k(N+1,\ell) = (-1)^{N}\sum_{q=0}^{\ell} S_q,$$

$$I(N+1,\ell) = (-1)^{N+1}\frac{p+N+1}{N+2}\binom{N+\ell+1}{\ell}\binom{p+N}{\ell+2N+3},$$

$$H_{N+1}(\ell) = (-1)^{N+1}\binom{\ell+N+1}{\ell}\binom{p+N}{\ell+2N+3}.$$

Consequently, by Lemma 3.1,

$$\sum_{k=0}^{N}(E_k(N+1,\ell)+F_{k+1}(N+1,\ell)) = I(N+1,\ell) = H_{N+1}(\ell)p.$$

This completes the proof of Lemma 3.16.

Lemma 3.17. *We have*

$$B(i)\beta^{-1}(i,1)P(1)\cdots P(N) = \tau(i+p-2,0)^{-p}Q_2(p-2)Q_3(p-2)$$

where

$$Q_2(p-2) = \prod_{k=0}^{p-3}\prod_{j=p-1}^{p-2+k}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{E_k(j,\ell)}$$

and

$$Q_3(p-2) = \prod_{k=1}^{p-2}\prod_{j=p-1}^{p-2+k}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{F_k(j,\ell)}.$$

To prove this take $N = p-2$ in Lemma 3.6. Then $Q_1(p-2)$ is an empty product and $H_{p-2}(0) = -1$.

Lemma 3.18. *We have*

$$B(i)\beta^{-1}(i,1)P(1)\cdots P(p-2) = \prod_{j=p-1}^{2p-4}\prod_{\ell=0}^{2p-4-j}\tau(i+j,\ell)^{S(j,\ell)}$$

where

$$S(j,\ell) = (-1)^{j-1}\binom{\ell+j-p+1}{j-p+1}\binom{j+1}{\ell+2(j-p)+5} + \sum_{k=j-p+2}^{p-3}E_k(j,\ell) + \sum_{k=j-p+2}^{p-2}F_k(j,\ell).$$

Proof. Apply (II) to $\tau(i+p-2,0)^{-p}$ in Lemma 3.17 and then gather like terms together.

Now the first term in the displayed equation for $S(j,\ell)$ can be included in the first sum by letting k run up to $p-2$ there. (See the definition of $E_k(j,\ell)$ given in Lemma 3.7.) So, by Lemma 3.7, $S(j,\ell) = 0$. Consequently, by Lemma 3.8,

$$\beta(i,1) = B(i)\prod_{k=1}^{p-2} P(k).$$

This, by the definition of $P(k)$, is the conclusion of Lemma 3.11.

4. We are now in a position to discuss the extension problem, i.e., prove that the groups under consideration do in fact exist. We shall assume that $\langle s, G_{i+1}\rangle$ is a p-group of maximal class, construct G_i as an extension of G_{i+1}, and then show that $\langle s, G_i\rangle$ is a p-group of maximal class. Our frame of reference is the known result: G is an extension of H with $|G\colon H| = p$ if and only if there is an x in G, an A in H, and an automorphism α of H such that $x^p = A$, $A^\alpha = A$, $h^{\alpha^p} = h^A$ for every h in H. This is proved in [3, p. 90].

Since $G_i = \langle s_i, G_{i+1}\rangle$ we shall, in the construction of G_i, take $x = s_i$, $s_i^p = A$, where A is defined by Lemma 1.3, and let $\alpha = s_i$ on G_{i+1}. That is, since

$$G_{i+1} = \{s_{i+1}^{c(i+1)}\cdots s_{n-1}^{c(n-1)}\colon 0 \le c(j) \le p-1, j = 1,\ldots,n-1\},$$

s_i is the mapping of G_{i+1} defined by

$$(s_{i+1}^{c(i+1)}\cdots s_{n-1}^{c(n-1)})^{s_i} = (s_{i+1}[s_{i+1},s_i])^{c(i+1)}\cdots(s_{n-1}[s_{n-1},s_i])^{c(n-1)}$$

where $[s_i,s_{i+1}]$ is defined by Lemma 1.2 and Theorem 1. Our first problem arises from the fact that s_i must be an automorphism of G_{i+1}.

Lemma 4.1. *Let s_i be the mapping of G_{i+1} defined above and suppose that s_i is an automorphism of G_{i+t+1}. Then s_i is an automorphism of G_{i+t} if and only if $(s_{i+t}^p)^{s_i} = (s_{i+t}^{s_i})^p$.*

Proof. Note first that our assumptions in Lemma 4.1 and Theorem 2 are drawn so that $[s_i,[x,y]] = 1$ for any x, y in G_{i+1}. Thus although G_i is not known to be a group at this stage we can, by an abuse of terminology, say that G_i is of class two. Secondly since s_i is, trivially, an automorphism of G_{n-1} we have a starting point for an induction.

So, suppose that s_i is an automorphism of G_{i+t+1}. Let $g = s_{i+t}^c x$ and $h = s_{i+t}^d y$ where $0 \le c, d \le p-1$, $c+d = kp+\ell$, $k = 0$ or 1 and x and y are in G_{i+t+1}. Since G_i is of class 2 and s_i is an automorphism of G_{i+t+1} we have

$$(gh)^{s_i} = (s_{i+t}^{s_i})^\ell (s_{i+t}^{kp})^{s_i}[x, s_{i+t}^d]^{s_i}(xy)^{s_i}$$

and

$$g^{s_i}h^{s_i} = (s_{i+t}^{s_i})^{kp}[x^{s_i},(s_{i+t}^{s_i})^d](xy)^{s_i}.$$

Consequently s_i is an automorphism of G_{i+t} if and only if $(s_{i+t}^p)^{s_i} = (s_{i+t}^{s_i})^p$ and $[x, s_{i+t}^d]^{s_i} = [x^{s_i}, (s_{i+t}^{s_i})^d]$. Lemma 4.1 follows from this for, since G_i is of class 2, the last equation is trivial.

Lemma 4.2. *Suppose the assumptions of Lemma* 4.1 *hold. Set*

$$A(i,t) = \prod_{u=0}^{p-1} [s_i, s_{i+t+u}]^{\binom{p}{u+1}}$$

for $t \geq 1$. *Then* $(s_{i+t}^{s_i})^p = (s_{i+t}^p)^{s_i}$ *if and only if* $A(i,t) = 1$.

Proof. First of all $(s_{i+t}^p)^{s_i} = (s_{i+t}^{s_i})^p$ if and only if $[s_i, s_{i+t}]^p = [s_i, s_{i+t}^p]$. This follows from the way s_i is defined on G_{i+1} and the fact that G_i is of class 2. Next, by Lemma 1.3 and by the assumption that s_i is an automorphism of G_{i+t+1},

$$[s_i, s_{i+t}^p] = [s_i, s_{i+t+1}^{-\binom{p}{2}} \cdots s_{i+t+p-1}^{-\binom{p}{p}}] = \prod_{u=1}^{p-1} [s_i, s_{i+t+u}]^{-\binom{p}{u+1}}.$$

Consequently, $[s_i, s_{i+t}]^p = [s_i, s_{i+t}^p]$ if and only if $A(i,t) = 1$.

Lemma 4.3. *Suppose the assumptions of Lemma* 4.1 *hold. Set*

$$\beta(i+1,t) = \prod_{a=1}^{p-1} \prod_{u+a}^{p-1} \lambda(i+1, t+a-1, u-a)^{-\binom{p}{u+1}}$$

for $t \geq 0$. *Then, for* $t \geq 1$, $A(i,t) = 1$ *if and only if* $\beta(i+1, t-1) = 1$.

Proof. To begin, for $t \geq 1$ and $u \geq 0$,

$$\lambda(i, t+u, 0) = \lambda(i,t,u)\lambda(i+1, t-1, u-1)^{-1}$$
$$\otimes \prod_{a=1}^{u} \lambda(i+1, t+a-1, u-a)^{-1} \lambda(i+1, t+a-1, u-a-1)^{-1},$$

where $\lambda(i,j,k) = 1$ for $k < 0$. This follows from Lemma 1.1 by induction on u. Thus

$$A(i,t) = \prod_{u=0}^{p-1} \lambda(i, t+u, 0)^{\binom{p}{u+1}} = P_1 P_2 P_3 P_4$$

where

$$P_1 = \prod_{u=0}^{p-1} \lambda(i,t,u)^{\binom{p}{u+1}},$$

$$P_2 = \prod_{u=1}^{p-1} \lambda(i+1, t-1, u-1)^{-\binom{p}{u+1}},$$

$$P_3 = \prod_{u=1}^{p-1} \prod_{a=1}^{u} \lambda(i+1, t+a-1, u-a)^{-\binom{p}{u+1}},$$

$$P_4 = \prod_{u=1}^{p-1} \prod_{a=1}^{u-1} \lambda(i+1, t+a-1, u-a-1)^{-\binom{p}{u+1}}.$$

Consider P_1. Since $\lambda(i, t, 0)$ is in an abelian group, say G_q, and since the relation $s_i^{\binom{p}{1}} \cdots s_{i+p-1}^{\binom{p}{p}} = 1$ for $i \geq q$ implies that

$$x^{\binom{p}{1}}[x, s]^{\binom{p}{2}} \cdots [x, \overbrace{s, \ldots, s}^{p-1}]^{\binom{p}{p}} = 1$$

for any x in G_q it follows that $P_1 = 1$.

Next, by definition $P_3 = \beta(i + 1, t)$.

Finally,

$$\begin{aligned} P_2 P_4 &= \prod_{u=1}^{p-1} \prod_{a=0}^{u-1} \lambda(i + 1, t + u - 1, u - a - 1)^{-\binom{p}{u+1}} \\ &= \prod_{u=1}^{p-1} \prod_{a=1}^{u} \lambda(i + 1, t - 2 + a, u - a)^{-\binom{p}{u+1}} = \beta(i + 1, t - 1). \end{aligned}$$

Consequently $A(i, t) = \beta(i + 1, t - 1)\beta(i + 1, t)$.

To continue, since s_i is an automorphism of G_{i+t+j} where $j \geq 1$ we have

$$\begin{aligned} A(i, t + 1) &= \beta(i + 1, t)\beta(i + 1, t + 1) = 1 \\ A(i, t + 2)^{-1} &= \beta(i + 1, t + 1)^{-1}\beta(i + 1, t + 2)^{-1} = 1 \\ &\cdot \\ &\cdot \\ &\cdot \\ A(i, t + j)^{(-1)^{j-1}} &= \beta(i + 1, t + j - 1)^{(-1)^{j-1}}\beta(i + 1, t + j)^{(-1)^{j-1}} = 1. \end{aligned}$$

Furthermore if j is sufficiently large ($j \geq n - 1 - i - t$) then $\beta(i + 1, t + j)$ is trivially equal to 1. Thus, since the right-hand side of the equations telescope, we have $\beta(i + 1, t) = 1$. But then, by the last equation of the last paragraph, $A(i, t) = \beta(i + 1, t - 1)$. This proves Lemma 4.3.

Lemma 4.4. *If* $\beta(i + 1, 1) = 1$ *then* $\beta(i + 1, t) = 1$ *for all* $t \geq 1$.

Proof. We have

$$\beta(i + 1, t) = \prod_{a=1}^{p-1} \prod_{u+a}^{p-1} \lambda(i + 1, t + a - 1, u - a)^{-\binom{p}{u+1}}.$$

In addition, for $t \geq 2$,

$$\begin{aligned} &\lambda(i + 1, t + a - 1, 0) \\ &\qquad = \lambda(i + 1, t + a - 2, 1)\lambda(i + 2, t + a - 3, 0)^{-1}\lambda(i + 2, t + a - 2, 0)^{-1}. \end{aligned}$$

Thus, for $t \geq 2$,

$$\beta(i+1,t) = [\beta(i+1,t-1),s]\beta(i+2,t-2)^{-1}\beta(i+2,t-1)^{-1}.$$

By induction (s_{i+1} is an automorphism of G_{i+2}), $\beta(i+2,\nu) = 1$ for all $\nu \geq 0$. Thus $\beta(i+1,t) = [\beta(i+1,t-1),s]$, which proves that if $\beta(i+1,1) = 1$ then $\beta(i+1,t) = 1$ for all $t \geq 1$.

We can now prove that s_i is an automorphism of G_{i+1}. Since $\langle s, G_{i+1}\rangle$ is a p-group of maximal class we have, by Lemma 1.6, $B(i+1) = 1$. Since $B(i+1) = 1$ we have, by Lemma 3.11, $\beta(i+1,1) = 1$. Thus, by Lemma 4.4, $\beta(i+1,t) = 1$ for all $t \geq 1$. Finally since $B(i+1) = \beta(i+1,1) = 1$ we can, by 2.3 Lemma and the discussion that follows its proof, choose the parameter $a(i+1,n-p+2)$ so that $\beta(i+1,0) = 1$. Since $\beta(i+1,0) = 1$, s_i is an automorphism of G_{i+1}.

Two other conditions arise in the construction of G_i from G_{i+1}: $A^{s_i} = A$ where $s_i^p = A$ and $h^{s_i^p} = h^A$ for every h in G_{i+1}.

As for the first of these, by Lemma 1.3, s_i^p is defined by

$$s_i^p = (s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}} q(i))^{-1} s_{n-1}^{x(i)}.$$

Thus, since G_i is of class 2, $A^{s_i} = A$ is equivalent to $[s_i, s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}}] = 1$. This is the relation "$B(i) = 1$" of Lemma 1.6.

The remaining condition $h^{s_i^p} = h^A$ for every h in G_{i+1} is easy to verify. For s_i is an automorphism of G_{i+1} and G_i is of class 2 so the condition under consideration reduces to

$$C(i,t) = \prod_{\nu=0}^{p-1} [s_{i+\nu}, s_{i+t}]^{\binom{p}{\nu+1}} = 1$$

for $t \geq 1$. (Take $h = s_{i+t}$ and use Lemma 1.3.) When $t = 1$ we have

$$C(i,1) = [s_i, s_{i+1}]^p \prod_{\nu=2}^{p-2} [s_{i+1}, s_{i+\nu}]^{-\binom{p}{\nu+1}}.$$

So, by Lemma 2.4, $C(i,1) = 1$.

If $t \geq 2$ we proceed as follows: By definition

$$C(i,t) = [s_i, s_{i+t}]^p \prod_{\nu=1}^{p-1} [s_{i+\nu}, s_{i+t}]^{\binom{p}{\nu+1}}. \tag{1}$$

Next, if $\langle s, G_i\rangle$ were a p-group of maximal class we would have

$$[s_i, s_{i+t}]^s = [s_i, s_{i+t}][s_i, s_{i+t+1}][s_{i+1}, s_{i+t}][s_{i+1}, s_{i+1+t}]. \tag{2}$$

Now $\langle s, G_i\rangle$ is not known to be a p-group of maximal class at this stage, but $\langle s, G_{i+1}\rangle$ is. Thus we can apply (2) to the commutators $[s_{i+\nu}, s_{i+t}]$ of (1).

Furthermore, by Lemmas 1.2 and then 2.5, $[s_i, s_{i+t}]^p$ can be expressed as a product of commutators of the form

$$[s_{i+k}, s_{i+k+l}, \overbrace{s, \ldots, s}^{j}]$$

where $k \geq 1$. Applying the $i + 1$ variant of (2) to these commutators we get, all told

$$C(i, 1)^s = C(i, 1)C(i, 2)C(i + 1, 1)$$

and

$$C(i, t)^s = C(i, t)C(i, t + 1)C(i + 1, t - 1)C(i + 1, t).$$

Since $C(i, 1) = 1$ it follows by induction on t that $C(i, t) = 1$ for $t \geq 1$.

This completes the construction of G_i from G_{i+1}.

Let us go on to the construction of $\langle s, G_i \rangle$. An argument similar to that given for Lemma 4.1 shows that s is an automorphism of G_i if and only if $(s_i^s)^p = (s_i^p)^s$ so we begin by examining this equation.

Lemma 4.5. *Let* $q(i)$ *and* $x(i)$ *be defined as in Lemma* 1.3. *Set*

$$P_1 = \prod_{l=0}^{p-1} s_{i+l}^{\binom{p}{l+1}} s_{i+l+1}^{\binom{p}{l+1}}$$

and

$$P_2 = \prod_{l=0}^{p-2} [s_{i+l}, s_{i+l+1}]^{g(l,1)}$$

where

$$g(l, 1) = -\binom{\binom{p}{l+1}}{2}.$$

Then $(s_i^s)^p = (s_i^p)^s$ *if and only if* $P_1 P_2 q(i)^s s_{n-1}^{-x(i)} = 1$.

Proof. According to Lemma 1.3,

$$s_i^p = (s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}} \quad q(i))^{-1} s_{n-1}^{x(i)}.$$

Consequently, since $s_i^s = s_i s_{i+1}$ and s is an automorphism of G_{i+1}, the equation $(s_i^s)^p = (s_i^p)^s$ is equivalent to

$$(1) \qquad (s_i s_{i+1})^{\binom{p}{1}} (s_{i+1} s_{i+2})^{\binom{p}{2}} \cdots (s_{i+p-1} s_{i+p})^{\binom{p}{p}} q(i)^s s_{n-1}^{-x(i)} = 1.$$

Next, by Theorem 4 of [5],

$$(2) \qquad (s_{i+l} s_{i+l+1})^{\binom{p}{l+1}} = s_{i+l}^{\binom{p}{l+1}} s_{i+l+1}^{\binom{p}{l+1}} [s_{u+l+1}, s_{i+l}]^{\binom{\binom{p}{l+1}}{2}}.$$

If we apply (2) to (1) and use the fact that G_i is of class 2 to move the commutators to the right we get Lemma 4.5.

Lemma 4.6. *Let P_1 be defined as in Lemma 4.5. Set*

$$R = \prod_{l=1}^{p-2} \prod_{t=1}^{p-1-l} [s_{i+l}, s_{i+l+t}]^{\binom{p}{l}\binom{p}{l+t+1}}.$$

Then $P_1 = q(i)^{-1} s_{n-1}^{x(i)} q(i+1)^{-1} R$. *Thus* $(s_i^s)^p = (s_i^p)^s$ *if and only if*

$$P_2 R[q(i), s] q(i+1)^{-1} = 1.$$

Proof. To start, rewrite P_1 as

$$P_1 = s_i^{\binom{p}{1}} (s_{i+1}^{\binom{p}{2}} s_{i+1}^{\binom{p}{1}}) (s_{i+2}^{\binom{p}{3}} s_{i+2}^{\binom{p}{2}}) Q_3 s_{i+p}$$

where

$$Q_3 = \prod_{l=3}^{p-1} (s_{i+l}^{\binom{p}{l+1}} s_{i+l}^{\binom{p}{l}}).$$

If we collect $s_{i+2}^{\binom{p}{3}}$ to the left and move the resulting commutator to the right we have

$$P_1 = (s_i^{\binom{p}{1}} s_{i+1}^{\binom{p}{2}} s_{i+2}^{\binom{p}{3}}) (s_{i+1}^{\binom{p}{1}} s_{i+2}^{\binom{p}{2}}) Q_3 [s_{i+1}^{\binom{p}{1}}, s_{i+2}^{\binom{p}{3}}].$$

By induction

$$P_1 = (s_i^{\binom{p}{1}} \ldots s_{i+r}^{\binom{p}{r+1}}) (s_{i+1}^{\binom{p}{1}} \ldots s_{i+r}^{\binom{p}{r}}) Q_{r+1} s_{i+p} R_{r-1}$$

where

$$Q_{r+1} = \prod_{l=r+1}^{p-1} (s_{i+l}^{\binom{p}{l+1}} s_{i+l}^{\binom{p}{l}})$$

and

$$R_{r-1} = \prod_{k=1}^{r-1} [s_{i+1}^{\binom{p}{1}} \ldots s_{i+k}^{\binom{p}{k}}, s_{i+k+1}^{\binom{p}{k+2}}].$$

So, by the above and then Lemma 1.3,

$$\begin{aligned} P_1 &= (s_i^{\binom{p}{1}} \ldots s_{i+p-1}{}^{\binom{p}{p}}) (s_{i+1}^{\binom{p}{1}} \ldots s_{i+p}^{\binom{p}{p}}) R_{p-2} \\ &= q(i)^{-1} s_{n-1}^{x(i)} q(i+1)^{-1} R_{p-2}. \end{aligned}$$

Setting $R = R_{p-2}$, this completes the proof of Lemma 4.6.

Lemma 4.7. *Let R be defined as in Lemma 4.6. Then*

$$R = \prod_{x=1}^{p-2} \prod_{y=0}^{p-2-x} \tau(i+x,y)^{e(x,y)}$$

where

$$e(x,y) = \sum_{j=0}^{x-1} (-1)^j \binom{y+j}{y}\binom{p}{x-j}\binom{p+j}{x+y+j+2}.$$

Proof. According to Lemma 1.2,

$$R = \prod_{\ell=1}^{p-2} \prod_{t=1}^{p-1-\ell} \prod_{j=0}^{w-1-i-\ell} \prod_{\omega=0}^{j} \tau(i+j+\ell, t-j-1-\omega)^{\binom{p}{\ell}\binom{p}{t+1+\ell}c(t,j,\omega)}.$$

To get Lemma 4.7 first hold ℓ and j fixed and gather together those terms where $t-j-1-\omega = y$. Then fix y and group together those terms where $j+\ell = x$. Note: in the second step one has $j+\ell = x$, so, since $\ell \geq 1$, the range of j is $0, \ldots, x-1$.

Lemma 4.8. *Let $T(i) = RP_2[q(i),s]q(i+1)^{-1}$ where R and P_2 are defined as above. Then*

$$T(i) = [\beta(i,0)\beta(i,1),s]\beta(i+1,0)^{-1}\beta(i+1,1)^{-1}B(i)^{-1}.$$

Proof. Let $\beta(i) = \beta(i,0)\beta(i,1)$ in what follows. In addition let

$$M_1(j,\ell) = \sum_{r=0}^{j} (-1)^r \binom{\ell+r}{r}\binom{p}{j-r}\binom{p+r}{\ell+j+2+r}$$

and

$$M_2(j,\ell) = \sum_{r=1}^{j+1} \binom{2r-1}{r}\binom{j+r}{j+1-r}\binom{p}{\ell+j+r+1}.$$

Recall, by Lemma 2.12, $M_1(j,\ell) = M_2(j,\ell)$.

Now by Lemmas 2.9 and 2.8 and Lemmas 4.5, 4.6 and 4.7,

$$[q(i),s] = [\beta(i),s]\prod_{\ell=1}^{p-1} \tau(i,\ell)^{-U_1(0,\ell-1)} \cdot \prod_{j=1}^{p-2}\prod_{\ell=1}^{p-1-j} \tau(i+j,\ell)^{-U_1(j,\ell-1)},$$

$$q^{-1}(i+1) = \beta(i+1)^{-1}\prod_{j=1}^{p-1} \tau(i+j,0)^{U_1(j-1,0)} \cdot \prod_{j=1}^{p-2}\prod_{\ell=1}^{p-1-j} \tau(i+j,\ell)^{U_1(j-1,\ell)},$$

$$R = \prod_{j=1}^{p-2} \tau(i+j,0)^{e(j,0)} \cdot \prod_{j=1}^{p-3}\prod_{\ell=1}^{p-2-j} \tau(i+j,\ell)^{e(j,\ell)},$$

$$P_2 = \prod_{j=0}^{p-2} \tau(i+j,0)^{g(j)},$$

where

$$U_1(j,\ell) = \sum_{r=1}^{j+1} \binom{2r-1}{r}\binom{r+j+1}{2r}\binom{p}{j+\ell+r+2},$$

$$e(j,\ell) = M_1(j,\ell) + (-1)^{j+1}\binom{\ell+j}{j}\binom{p+j}{\ell+2j+2},$$

$$g(j) = -\binom{\binom{p}{j+1}}{2}.$$

Let us consider the total exponent of $\tau(i+j,\ell)$ in the above scheme when $j \geq 1$ and $1 \leq \ell \leq p-1-j$. We have

$$U_1(j-1,\ell) - U_1(j-1,\ell) = -M_2(j,\ell).$$

Consequently,

$$U_1(j-1,\ell) - U_1(j-1,\ell) + e(j,\ell) = (-1)^{j+1}\binom{\ell+j}{j}\binom{p+j}{\ell+2j+2}.$$

Consider next the exponent of $\tau(i+j,0)$ where $j \geq 1$. We have

$$U_1(j-1,0) = \sum_{r=1}^{j} \binom{2r-1}{r}\binom{r+j}{2r}\binom{p}{j+r+1}$$

$$\text{and } g(j) = -\binom{\binom{p}{j+1}}{2}.$$

So, by a trivial variation of Lemma 2.13,

$$U_1(j-1,0) + g(j) = -M_2(j,0).$$

Since $e(j,0) = M_2(j,0) + (-1)^{j+1}\binom{p+j}{2j+2}$, it follows that

$$U_1(j-1,0) + g(j) + e(j,0) = (-1)^{j+1}\binom{p+j}{2j+2}.$$

The product

$$\tau(i,0)^{g(0)} \prod_{\ell=1}^{p-1} \tau(i,\ell)^{-U_1(0,\ell-1)}$$

remains to be considered. But $g(0) = -\binom{p}{2}$ and $-U(0,\ell-1) = -\binom{p}{\ell+2}$.

Bringing these results together and referring to the double product representation of $B(i)$ in Lemma 1.6 we get the conclusion of Lemma 4.8.

Lemma 4.8 implies that $T(i) = 1$. For we have $B(i) = \beta(i+1,0) = \beta(i+1,1) = 1$ from our earlier work. Furthermore, by Lemma 3.11 if $B(i) = 1$ then $\beta(i,1) = 1$ and, by Lemma 2.3,

$$[\beta(i,0),s]\beta(i,1)^{-1} = B(i).$$

Consequently $[\beta(i,0)\beta(i,1),s] = 1$, so $T(i) = 1$, and s is an automorphism of G_i.

The two remaining construction conditions are: If $s^p = A$ then $A^s = A$ and $h^{s^p} = h^A$ for every h in G_i. We have $s^p = s_{n-1}^x$ where s_{n-1} is in the center of G so $A = s_{n-1}^x$ and $A^s = A$ is trivial. Moreover, since $\langle s, G_{i+1}\rangle$ is a p-group of maximal class the condition $h^{s^p} = h^A$ reduces to $s_i^{s^p} = s_i$. This fact follows from

Lemma 4.9. *Let $q(i)$ be defined as in Lemma* 1.3 *and $r(i)$ as in Lemma* 1.4. *Then $s_i^{s^p} = s_i$ if and only if $q^{-1}(i+1)r(i+1) = 1$.*

Proof. If we take $x = s$ and $y = s_i$ in Theorem 3 of [5] we get (see the proof of Lemma 1.4)

$$s_i^{s^p} = s_i s_{i+1}^{\binom{p}{1}} \cdots s_{i+p}^{\binom{p}{p}} r(i+1).$$

Thus, by Lemma 1.3, $s_i^{s^p} = s_i$ if and only if $q^{-1}(i+1)r(i+1)$. This completes the proof of Lemma 4.9.

By Lemma 2.12, $q^{-1}(i+1)r(i+1) = \beta(i+1,0)^{-1}\beta(i+1,1)^{-1}$. This verifies the condition and completes the construction of $\langle s, G_i\rangle$.

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Department of Mathematics, University of California, Los Angeles, California 90024